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Properties of the Principal Eigenvalues of a General Class of Non-classical Mixed Boundary Value Problems

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In this paper we characterize the existence of principal eigenvalues for a general class of linear weighted second order elliptic boundary value problems subject to a very general class of mixed boundary conditions. Our theory is a substantial extension of the classical theory by P. Hess and T. Kato (1980, *Comm. Partial Differential Equations* **5**, 999–1030). In obtaining our main results we must give a number of new results on the continuous dependence of the principal eigenvalue of a second order linear elliptic boundary value problem with respect to the underlying domain and the boundary condition itself. These auxiliary results complement and in some sense complete the theory of D. Daners and E. N. Dancer (1997, *J. Differential Equations* **138**, 86–132). The main technical tool used throughout this paper is a very recent characterization of the strong maximum principle in terms of the existence of a positive strict supersolution due to H. Amann and J. López-Gómez (1998, *J. Differential Equations* **146**, 336–374).

1. INTRODUCTION

In this paper we study the existence, multiplicity, and main properties of the principal eigenvalues of the linear weighted boundary value problem

$$\begin{cases} \mathscr{L}\varphi = \lambda W(x)\,\varphi & \text{in} \quad \Omega, \\ \mathscr{B}(b)\,\varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

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where we make the following assumptions:

(a) Ω is a bounded domain in $\mathbf{R}^N$, $N \geqslant 1$, of class $\mathscr{C}^2$, i.e., $\bar{\Omega}$ is an N-dimensional compact connected $\mathscr{C}^2$-submanifold of $\mathbf{R}^N$ with boundary $\partial\Omega$ of class $\mathscr{C}^2$.

(b) $\lambda \in \mathbf{R}$, $W \in L_\infty(\Omega)$ and

$$\mathscr{L} := -\sum_{i,j=1}^{N} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i(x) \frac{\partial}{\partial x_i} + \alpha_0(x) \tag{1.2}$$

is uniformly strongly elliptic in Ω with

$$\alpha_{ij} = \alpha_{ji} \in \mathscr{C}(\bar{\Omega}), \qquad \alpha_k \in L_\infty(\Omega), \qquad 1 \leqslant i, j \leqslant N, \quad 0 \leqslant k \leqslant N. \tag{1.3}$$

In the sequel we denote by $\mu > 0$ the ellipticity constant of $\mathscr{L}$ in Ω. Then, for any $\xi \in \mathbf{R}^N \setminus \{0\}$ and $x \in \bar{\Omega}$ we have that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x)\, \xi_i \xi_j \geqslant \mu\, |\xi|^2.$$

(c) $\mathscr{B}(b)$ stands for the boundary operator

$$\mathscr{B}(b)\,\varphi := \begin{cases} \varphi & \text{on} \quad \Gamma_0, \\ \partial_\nu \varphi + b\varphi & \text{on} \quad \Gamma_1, \end{cases} \tag{1.4}$$

where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $b \in \mathscr{C}(\Gamma_1)$,

$$\nu = (\nu_1, ..., \nu_N) \in \mathscr{C}^1(\Gamma_1, \mathbf{R}^N)$$

is an outward pointing nowhere tangent vector field, and

$$\partial_\nu \varphi := \langle \nabla\varphi, \nu \rangle.$$

Moreover, Γ_0 and Γ_1 possess finitely many components. Thus, $\mathscr{B}(b)$ is the Dirichlet boundary operator on Γ_0, denoted in the sequel by $\mathscr{D}$, and the Neumann or a first order regular oblique derivative boundary operator on Γ_1. It should be pointed out that either Γ_0 or Γ_1 may be empty.

By a principal eigenvalue we mean a value of $\lambda \in \mathbf{R}$ for which there exists a positive φ satisfying the linear boundary value problem (1.1).

The problem of analyzing the existence of principal eigenvalues for (1.1) is absolutely necessary for a complete understanding of the structure of the set of positive solutions of wide classes of semilinear elliptic boundary value problems of the form

$$\begin{cases} \mathscr{L}u = \lambda f(x,u)\,u & \text{in} \quad \Omega, \\ \mathscr{B}(b)\,u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where $f\colon \bar{\Omega}\times[0,\infty)\to \mathbf{R}$ is a continuous function and λ is regarded as a real parameter. Note that λ can be thought as the inverse of a diffusion coefficient $d := 1/\lambda$ in front of $\mathscr{L}$ when $\lambda > 0$, and that from the point of view of the applications to the applied sciences and engineering one is interested in analyzing how varies the dynamics of the positive solutions of the parabolic model associated to (1.5) as the diffusion, or equivalently λ, changes. Adopting this point of view the principal eigenvalues of the linearization around any positive solution of the problem provide us with the ranges of values of the parameter for which the mathematical solution can be a physical solution of a system governed by (1.5).

As far as to the mathematics contained in this paper concerns, it should be pointed out that our general boundary conditions do not fit into the classical setting, since we are dealing with mixed boundary conditions and b might be negative on some region of some of the components of Γ_1. In applications these boundary conditions arise in a natural way; for example, when one linearizes around a positive solution of the nonlinear radiation boundary value problem

$$\begin{cases} \mathscr{L}u = \lambda f(x,u)\,u & \text{in} \quad \Omega, \\ \partial_\nu u = u^p & \text{on} \quad \partial\Omega, \end{cases} \tag{1.6}$$

where $p > 1$ and $f\colon \bar{\Omega}\times[0,\infty)\to \mathbf{R}$ is a function of class $\mathscr{C}^1$. Indeed, if u_0 is a positive solution of (1.6), then the linearization of (1.6) around it is given by

$$\begin{cases} \mathscr{L}u = \lambda[\partial_u f(x,u_0(x))\,u_0(x) + f(x,u_0(x))]\,u & \text{in} \quad \Omega, \\ \partial_\nu u - p u_0^{p-1}(x)\,u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

and hence, by making the choice

$$\begin{aligned} W(x) &:= \partial_u f(x,u_0(x))\,u_0(x) + f(x,u_0(x)), \\ b(x) &:= -p u_0^{p-1}(x), \qquad \Gamma_0 = \emptyset, \end{aligned} \tag{1.8}$$

(1.7) fits into our abstract setting. Note that $b<0$ on Γ_1. The principal eigenvalues of (1.8) provide us with the values of the parameter λ where the solution u_0 becomes stable, and therefore their knowledge is absolutely crucial in order to predict the behavior of these nonlinear radiation problems in the large.

The analysis of the classical case when the potential W possesses definite sign and it is bounded away from zero does not entail any special mathematical difficulty and it is very well documented in the literature (e.g., Courant and Hilbert [11]); in the classical case (1.1) possesses a unique principal eigenvalue. The analysis of (1.1) in the most general and interesting situation when the potential W changes of sign goes back to the pioneer works of Manes and Micheletti [31], where the special case when $\mathscr{L}$ is selfadjoint was dealt with, and Hess and Kato [22] (cf. Hess [21] and the references therein), where for non-necessarily selfadjoint operators it was found that if the coefficients of $\mathscr{L}$ and the potential W are Hölder continuous, and in addition $b \geqslant 0$, v is the outward unit normal, $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0$, and either $\Gamma_0 = \emptyset$, or $\Gamma_1 = \emptyset$, then (1.1) possesses two principal eigenvalues; one of them negative and the other positive. Hereafter, $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ will stand for the principal eigenvalue of $\mathscr{L}$ in Ω subject to the boundary operator $\mathscr{B}(b)$ defined by (1.4), that is to say, $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ will stand for the only eigenvalue of the problem

$$\begin{cases} \mathscr{L}\varphi = \sigma\varphi & \text{in} \quad \Omega, \\ \mathscr{B}(b)\,\varphi = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

possessing a positive eigenfunction. Later, one of the authors of this paper removed the coercivity requirement

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0 \tag{1.9}$$

from the statement of the theorem by P. Hess and T. Kato in the very special case when $\Gamma_1 = \emptyset$ (cf. [28, 29]), i.e., when $\mathscr{B}(b) = \mathscr{D}$ is the Dirichlet operator. Such a substantial generalization was possible thanks the following result.

THEOREM 1.1. *Suppose $W \geqslant 0$, $W \neq 0$, is a nice potential having a nice vanishing set*

$$\Omega_0 := \Omega \setminus \overline{\{x \in \Omega : W(x) > 0\}}.$$

Then,

$$\lim_{\lambda \searrow -\infty} \sigma_1^\Omega[\mathscr{L} - \lambda W, \mathscr{D}] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{D}]. \tag{1.10}$$

Therefore, since

$$\lim_{\lambda \nearrow \infty} \sigma_1^{\Omega}[\mathscr{L} - \lambda W, \mathscr{D}] = -\infty$$

and the map

$$\lambda \to \sigma_1^{\Omega}[\mathscr{L} - \lambda W, \mathscr{D}]$$

is decreasing, it follows that (1.1) possesses a principal eigenvalue if, and only if,

$$\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{D}] > 0. \tag{1.11}$$

Thanks to a celebrated inequality due to Faber [16] and Krahn [27], (1.11) is satisfied if the Lebesgue measure of Ω_0, $|\Omega_0|$, is small enough; independently of the sign of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{D}]$. Note that the previous characterization of the existence of a principal eigenvalue under the assumptions of Theorem 1.1 is rather natural since W approaches to a *classical potential* as $|\Omega_0| \searrow 0$. Now, thanks to the continuity of the principal eigenvalue with respect to the potential, one can easily realize that under condition (1.11) if W is perturbed by an small amplitude potential $\tilde{W} < 0$ then the linear problem (1.1) for the perturbed potential $W + \tilde{W}$ should have at least one principal eigenvalue close to the principal eigenvalue of the unperturbed problem. Further, using the analyticity and concavity of the map

$$\lambda \to \Sigma(\lambda) := \sigma_1^{\Omega}[\mathscr{L} - \lambda(W + \tilde{W}), \mathscr{D}]$$

shows the existence of exactly two principal eigenvalues [29]. Summarizing, if (1.11) is satisfied and the amplitude of $\tilde{W} < 0$ is small enough, then the problem (1.1) for the perturbed potential $W + \tilde{W}$ possesses two principal eigenvalues; the two zeros of the map $\Sigma(\lambda)$.

It is worth mentioning that, beside the interest of Theorem 1.1 to characterize the existence of principal eigenvalues of (1.1), (1.10) has shown to be crucial in the semi-classical analysis of second order elliptic operators involving potentials with degenerate wells. Actually, Theorem 1.1 was the starting point to analyze some old open problems proposed by Simon [35] (cf. Dancer and J. López-Gómez [15]).

In this paper we adopt the same methodology as in [29]. Thus, our first goal will be obtaining a sharp general version of Theorem 1.1 in our general abstract setting. Actually, the results of this paper provide us with substantial improvements of all the results found in [29], even in the simplest case when $\mathscr{B}(b)$ is the Dirichlet operator, $\mathscr{D}$. But now live will not be so easy as it was in [29], since Theorem 1.1 is based upon the continuous dependence of $\sigma_1^\Omega[\mathscr{L}, \mathscr{D}]$ with respect to Ω and it turns out that working under our general boundary conditions the continuous dependence of $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ is known to fail if $\Gamma_0 = \emptyset$ and $b = 0$ (cf. the counterexample of Section VI.2.6 in Volume I of [11]). As a result of this failure, a huge effort has been made in analyzing how varies the spectrum of $-\Delta$ under homogeneous Neumann boundary conditions when the domain is perturbed, e.g., Hale and Vegas [19], Arrieta [5], Arrieta *et al.* [6], and Jimbo [23, 24], where a precise analysis of the behavior of the Neumann eigenvalues for dumbbell like domains with handle shrinking to a segment was done. Later, Dancer and Daners [14] showed how the classical Robin problem, i.e., the case when $\Gamma_0 = \emptyset$ and b is bounded bellow by a positive constant, behaves much like the pure Dirichlet problem, basically because the classical Robin problems have smoothing properties similar to the Dirichlet problem, independently of the geometry of the domain, whereas this is not the case for the Neumann problem. Some earlier results on Robin boundary value problems can be found in Ward and Keller [37] and Ward *et al.* [38], where the method of matched asymptotic expansions was used to calculate the perturbed eigenvalues and eigenfunctions for some special cases of interest in the applications. Since in this paper we are working under general mixed boundary conditions and in particular $b(x)$ is allowed to vanish on some piece of some of the components of Γ_1, being in addition negative on other pieces of these components, to get the continuous dependence of $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ with respect to Ω we must restrict ourselves to consider perturbations of Ω around its Dirichlet boundary, Γ_0.

Under our general assumptions the existence and the uniqueness of the principal eigenvalue $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ goes back to Amann [3], and the characterization of the strong maximum principle in terms of the positivity of the principal eigenvalue and in terms of the existence of a positive strict supersolution is a very recent result coming from Amann and López-Gómez [4], where the previous characterization of López-Gòmez and Molina-Meyer [30], found originally for Dirichlet boundary conditions, was shown to be satisfied for a general boundary operator of the form (1.4). Such characterization is the key technical tool to get most of the comparison results used throughout this work, among them the monotonicities of $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ with respect to the potential and the underlying domain, its point-wise min-max characterization, and its concavity.

The continuous dependence of $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ with respect to the perturbations of Ω around Γ_0 will provide us with the following very deep and substantially sharper counterpart of Theorem 1.1.

THEOREM 1.2. *Suppose*

$$\alpha_{ij} \in \mathscr{C}^1(\bar{\Omega}), \qquad \alpha_i \in \mathscr{C}(\bar{\Omega}), \qquad 1 \leqslant i, j \leqslant N,$$

and $W \in L_\infty(\Omega)$, $W \geqslant 0$, is a potential for which there exist an open subset Ω_0 of Ω and a compact subset K of $\bar{\Omega}$ with Lebesgue measure zero such that

$$K \cap (\bar{\Omega}_0 \cup \Gamma_1) = \emptyset,$$
$$\Omega_+ := \{x \in \Omega : W(x) > 0\} = \Omega \setminus (\bar{\Omega}_0 \cup K),$$

and each of the following conditions is satisfied:

(a) *Ω_0 possesses a finite number of components of class $\mathscr{C}^2$, say Ω_0^j, $1 \leqslant j \leqslant m$, such that $\bar{\Omega}_0^i \cap \bar{\Omega}_0^j = \emptyset$ if $i \neq j$, and*

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0.$$

Thus, if we denote by Γ_1^i, $1 \leqslant i \leqslant n_1$, the components of Γ_1, then for each $1 \leqslant i \leqslant n_1$ either $\Gamma_1^i \subset \partial\Omega_0$ or $\Gamma_1^i \cap \partial\Omega_0 = \emptyset$. Moreover, if $\Gamma_1^i \subset \partial\Omega_0$, then Γ_1^i must be a component of $\partial\Omega_0$. Indeed, if $\Gamma_1^i \cap \partial\Omega_0 \neq \emptyset$ but Γ_1^i is not a component of $\partial\Omega_0$, then $\operatorname{dist}(\Gamma_1^i, \partial\Omega_0 \cap \Omega) = 0$.

(b) *Let $\{i_1, \ldots, i_p\}$ denote the subset of $\{1, \ldots, n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_0 = \emptyset$ if and only if $j \in \{i_1, \ldots, i_p\}$. Then, W is bounded away from zero on any compact subset of*

$$\Omega_+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}.$$

(c) *Let Γ_0^i, $1 \leqslant i \leqslant n_0$, denote the components of Γ_0, and let $\{i_1, \ldots, i_q\}$ be the subset of $\{1, \ldots, n_0\}$ for which $(\partial\Omega_0 \cup K) \cap \Gamma_0^j \neq \emptyset$ if and only if $j \in \{i_1, \ldots, i_q\}$. Then, W is bounded away from zero on any compact subset of*

$$\Omega_+ \cup \left[\bigcup_{j=1}^{q} \Gamma_0^{i_j} \setminus (\partial\Omega_0 \cup K) \right].$$

(d) *For any $\eta > 0$ there exist a natural number $\ell(\eta) \geqslant 1$ and $\ell(\eta)$ open subsets of $\mathbf{R}^N$, G_j^η, $1 \leqslant j \leqslant \ell(\eta)$, with $|G_j^\eta| < \eta$, $1 \leqslant j \leqslant \ell(\eta)$, such that*

$$\bar{G}_i^\eta \cap \bar{G}_j^\eta = \emptyset \qquad \text{if} \quad i \neq j,$$
$$K \subset \bigcup_{j=1}^{\ell(\eta)} G_j^\eta,$$

and for each $1 \leqslant j \leqslant \ell(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathscr{C}^2$.

(e) *ν is the conormal field on $\Gamma_1 \cap \partial\Omega_0$.*

Then,

$$\lim_{\lambda \nearrow \infty} \sigma_1^\Omega[\mathscr{L} + \lambda W, \mathscr{B}(b)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)], \tag{1.12}$$

where $\mathscr{B}(b, \Omega_0)$ *is the boundary operator defined by*

$$\mathscr{B}(b, \Omega_0)\, \varphi := \begin{cases} \varphi & \text{on} \quad \partial\Omega_0 \cap \Omega, \\ \mathscr{B}(b)\, \varphi & \text{on} \quad \partial\Omega_0 \cap \partial\Omega. \end{cases}$$

The proof of Theorem 1.2 is based upon the characterization of the strong maximum principle of Amann and López-Gómez [4] through by the construction of adequate supersolutions; such construction is very delicate since it contains several fine technical details. During the construction of these supersolutions we must slightly enlarge Ω_0 and this is the precise moment when the continuous dependence of the principal eigenvalue with respect to the domain is needed.

The continuous dependence will be shown to be true for a very general class of domains which are stable with respect to the boundary operator $\mathscr{B}(b)$ (cf. Section 6 for further details). It seems that this is the first work where the concept of stability has been introduced in the context of general boundary operators. The concept of stability of a domain goes back to Babuška [7] and Babuška and Vyborny [8] where it was used to generalize some pioneer results of Courant and Hilbert [11] on the continuous variation of principal eigenvalues with respect to the domain Ω in the special case when $\Gamma_1 = \emptyset$ and $\mathscr{L}$ is assumed to be selfadjoint. Stability of a subset of $\mathbf{R}^N$ is a very mild condition that, beside playing a central rol in potential theory since it provides us with all domains for which the Dirichlet problem makes sense (cf. Adams and Hedberg [1], Dancer [13], and the references there in), has shown to govern other important problems in analysis (cf. Hedberg [20]).

Another interesting property that we are going to analyze in this paper is the continuous dependence of the principal eigenvalue $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ with respect to the boundary weight function b. Our main result reads as follows.

THEOREM 1.3. *Let* $\sigma(L_\infty(\Gamma_1), L_1(\Gamma_1))$ *denote the weak* $*$ *topology of* $L_\infty(\Gamma_1)$ *and assume that* $b_n \in \mathscr{C}(\Gamma_1)$, $n \geqslant 1$, *is a sequence such that*

$$\lim_{n\to\infty} b_n = b \qquad \text{in} \quad \sigma(L_\infty(\Gamma_1), L_1(\Gamma_1)).$$

Then

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_n)] = \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)].$$

It seems that this is the first general result concerning with the continuous dependence of the principal eigenvalue with respect to b available in the literature. It should be pointed out that Theorem 1.3 is a very sharp result having very strong consequences; obviously out of the scope of this work. For instance, it readily follows from Theorem 1.3 that if u_0 is a positive stable solution of (1.6) and u_1 is a positive unstable solution of (1.6), then u_0 and u_1 must be bounded away in the L_∞ norm.

In order to construct general families of indefinite potentials for which (1.1) possesses two principal eigenvalues we will decompose the potential into the difference between its positive and its negative parts

$$W = W^+ - W^-, \qquad W^+ := \max\{W, 0\},$$

and will assume that, for instance, W^+ satisfies all the requirements of Theorem 1.2 and

$$\sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{B}(b, \Omega_0^+)] > 0, \tag{1.13}$$

where Ω_0^+ stands for the maximal vanishing open set of W^+. So, a rather natural question arises. How should be Ω_+ and b to get the coercivity condition (1.13)?. In the case when $\mathscr{B}(b, \Omega_0^+) = \mathscr{D}$ we have found that $\sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{D}]$ grows to infinity with order $|\Omega_0^+|^{-\frac{2}{N}}$ as $|\Omega_0^+| \searrow 0$. Precisely, we have obtained the following result.

THEOREM 1.4. *Suppose*

$$\alpha_{ij} \in \mathscr{C}(\bar{\Omega}) \cap W^1_\infty(\Omega), \qquad 1 \leqslant i, j \leqslant N.$$

Then

$$\liminf_{|\Omega_0^+| \searrow 0} \sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{D}]\, |\Omega_0^+|^{\frac{2}{N}} \geqslant \mu \Sigma_1\, |B_1|^{\frac{2}{N}},$$

where B_1 is the unit ball of $\mathbf{R}^N$, $\Sigma_1 = \sigma_1^{B_1}[-\Delta, \mathscr{D}]$ and μ is the ellipticity constant of $\mathscr{L}$.

On the other hand, it turns out that the following result is satisfied.

THEOREM 1.5. *Suppose*

$$\alpha_{ij} \in \mathscr{C}^1(\bar{\Omega}), \qquad \alpha_i \in \mathscr{C}(\bar{\Omega}), \qquad 1 \leqslant i, j \leqslant N,$$

and let $b_n \in \mathscr{C}(\Gamma_1 \cap \partial\Omega_0^+)$, $n \geqslant 1$, be an arbitrary sequence such that

$$\lim_{n\to\infty} \min_{\Gamma_1 \cap \partial\Omega_0^+} b_n = \infty.$$

Then

$$\lim_{n \to \infty} \sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{B}(b_n, \Omega_0^+)] = \sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{D}].$$

Thus, combining Theorem 1.4 with Theorem 1.5 we find that (1.13) is satisfied if $|\Omega_0^+|$ is sufficiently small and $b|_{\Gamma_1 \cap \partial\Omega_0^+}$ is sufficiently large, which completely answers to the question raised above.

Up to now we have discussed some of the main results of this paper. We now shortly describe its organization. In Section 2 we include some preliminaries, among them the characterization of the strong maximum principle found in [4], and set the most common notations used throughout the paper. In Section 3 we use the characterization of the strong maximum principle to obtain the monotonicities of the principal eigenvalue with respect to the potential, the domain, and the boundary weight function b. In Section 4 we use the characterization of the strong maximum principle to give a point-wise min-max characterization of the principal eigenvalue which extends a previous result by Protter and Weinberger [33] and Theorem 4.5 of Pinsky [32], found for Dirichlet boundary conditions. In Section 5 we combine the min-max characterization found in Section 4 with the ellipticity of $\mathscr{L}$ to give an elementary proof of the concavity of the principal eigenvalue with respect to the potential. Though our result is substantially sharper, we should mention that the concavity of the spectral bound and type with respect to the potential goes back to Kato [25]. The concavity will be use further, in Section 12, to get the exact multiplicity results for the principal eigenvalues of (1.1). In Section 6 we introduce the concept of stability of a domain for our general boundary conditions and show that any domain satisfying the segment property is stable. In Section 7 we prove the continuous dependence of the principal eigenvalue with respect to any admissible perturbation of an stable domain. Even when we restrict ourselves to consider the special case when $\Gamma_1 = \emptyset$ our results are substantial improvements of the corresponding results of [29], since in this paper we are assuming that

$$\alpha_{ij} \in \mathscr{C}^1(\bar{\Omega}), \qquad \alpha_i \in \mathscr{C}(\bar{\Omega}), \qquad 1 \leqslant i, j \leqslant N,$$

instead of

$$\alpha_{ij} \in \mathscr{C}^2(\bar{\Omega}), \qquad \alpha_i \in \mathscr{C}^1(\bar{\Omega}), \qquad 1 \leqslant i, j \leqslant N,$$

as it was assumed in [29] (cf. Theorem 4.2 of [29]). In Section 8 we prove Theorem 1.3, in Section 9 we prove Theorem 1.5, in Section 10 we prove Theorem 1.4, and in Section 11 we prove Theorem 1.2. Finally, in Section 12 we will apply the previous theory to characterize the existence of principal eigenvalues of (1.1).

2. PRELIMINARIES AND NOTATIONS

Under the assumptions of Section 1 for each $p > 1$ we denote

$$W^2_{p,\mathscr{B}(b)}(\Omega) := \{u \in W^2_p(\Omega) : \mathscr{B}(b)\, u = 0\},$$

and

$$W^2_{\mathscr{B}(b)}(\Omega) := \bigcap_{p>1} W^2_{p,\mathscr{B}(b)}(\Omega) \subset H^2(\Omega).$$

Also, in the sequel we use the natural product order on $L_p(\Omega) \times L_p(\partial\Omega)$. Namely,

$$(f_1, g_1) \geqslant (f_2, g_2) \Leftrightarrow f_1 \geqslant f_2 \wedge g_1 \geqslant g_2 .$$

It will be said that $(f_1, g_1) > (f_2, g_2)$ if $(f_1, g_1) \geqslant (f_2, g_2)$ and $(f_1, g_1) \neq (f_2, g_2)$.

Since $b \in \mathscr{C}(\Gamma_1)$, we have from the results of [3] that for each $p > 1$

$$\mathscr{B}(b) \in \mathscr{L}(W^2_p(\Omega), W_p^{2-\frac{1}{p}}(\Gamma_0) \times W_p^{1-\frac{1}{p}}(\Gamma_1)).$$

Moreover, there exists a least real eigenvalue of the problem

$$\begin{cases} \mathscr{L}\varphi = \lambda\varphi & \text{in } \Omega, \\ \mathscr{B}(b)\,\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

denoted by $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ and called *principal eigenvalue of* $(\mathscr{L}, \mathscr{B}(b), \Omega)$. The principal eigenvalue is simple and associated with it there is a positive eigenfunction, unique up to multiplicative constants, denoted by $\varphi_{[\mathscr{L},\mathscr{B}(b)]}$ and called *principal eigenfunction of* $(\mathscr{L}, \mathscr{B}(b), \Omega)$. Thanks to Theorem 12.1 of [3] the principal eigenfunction satisfies

$$\varphi_{[\mathscr{L},\mathscr{B}(b)]} \in W^2_{\mathscr{B}(b)}(\Omega) \subset H^2(\Omega)$$

and it is *strongly positive in* Ω, in the sense that

$$\varphi_{[\mathscr{L},\mathscr{B}(b)]}(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \wedge \quad \partial_\nu \varphi_{[\mathscr{L},\mathscr{B}(b)]}(x) < 0 \quad \forall\, x \in \Gamma_0 .$$

In fact, $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ is the only eigenvalue of (2.1) possessing a positive eigenfunction, and it is dominant in the sense that any other eigenvalue σ of (2.1) satisfies

$$\Re\sigma > \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)].$$

Furthermore, setting

$$\mathscr{L}_p := \mathscr{L}|_{W^2_{p,\mathscr{B}(b)}(\Omega)},$$

it turns out that for each $\omega > -\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ and $p > N$ the operator

$$(\omega + \mathscr{L}_p)^{-1} \in \mathscr{L}(L_p(\Omega))$$

is positive, compact, and irreducible.

Given any proper subdomain Ω_0 of Ω of class $\mathscr{C}^2$ with

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0 \tag{2.2}$$

we shall denote by $\mathscr{B}(b, \Omega_0)$ the boundary operator build up from $\mathscr{B}(b)$ by

$$\mathscr{B}(b, \Omega_0)\,\varphi := \begin{cases} \varphi & \text{on} \quad \partial\Omega_0 \cap \Omega, \\ \mathscr{B}(b)\,\varphi & \text{on} \quad \partial\Omega_0 \cap \partial\Omega. \end{cases} \tag{2.3}$$

When $\Omega_0 = \Omega$ we set

$$\mathscr{B}(b, \Omega) := \mathscr{B}(b).$$

If $\bar{\Omega}_0 \subset \Omega$, then $\partial\Omega_0 \subset \Omega$ and by definition

$$\mathscr{B}(b, \Omega_0)\,\varphi = \varphi,$$

i.e., $\mathscr{B}(b, \Omega_0)$ becomes into the Dirichlet boundary operator, $\mathscr{D}$. Also, we denote by $\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)]$ the principal eigenvalue of the linear boundary value problem

$$\begin{cases} \mathscr{L}\varphi = \lambda\varphi & \text{in} \quad \Omega_0, \\ \mathscr{B}(b, \Omega_0)\,\varphi = 0 & \text{on} \quad \partial\Omega_0. \end{cases} \tag{2.4}$$

Suppose $p > N$. Then, a function $\bar{u} \in W_p^2(\Omega)$ is said to be a *positive strict supersolution* of $(\mathscr{L}, \mathscr{B}(b), \Omega)$ if $\bar{u} \geqslant 0$ and $(\mathscr{L}\bar{u}, \mathscr{B}(b)\,\bar{u}) > 0$. A function $u \in W_p^2(\Omega)$ is said to be *strongly positive* if $u(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial_\beta u(x) < 0$ for each $x \in \Gamma_0$ with $u(x) = 0$ and any outward pointing nowhere tangent vector field $\beta \in \mathscr{C}^1(\Gamma_0, \mathbf{R}^N)$. Finally, $(\mathscr{L}, \mathscr{B}(b), \Omega)$ is said to satisfy the *strong maximum principle* if $p > N$, $u \in W_p^2(\Omega)$, and $(\mathscr{L}u, \mathscr{B}(b)\,u) > 0$ imply that u is strongly positive. Recall that for any $p > N$

$$W_p^2(\Omega) \hookrightarrow \mathscr{C}^{2-\frac{N}{p}}(\bar{\Omega})$$

and that each $u \in W_p^2(\Omega)$ is a.e. in Ω twice differentiable (cf. Theorem VIII.1 of [36]).

The following characterization of the strong maximum principle provides us with one of the main technical tools to obtain most of the results of this paper. It was found in [4], where some former weaker versions given in [29, 30] were generalized.

THEOREM 2.1. *The following assertions are equivalent:*

(a) $\sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)] > 0$;

(b) $(\mathcal{L}, \mathcal{B}(b), \Omega)$ *possesses a positive strict supersolution;*

(c) $(\mathcal{L}, \mathcal{B}(b), \Omega)$ *satisfies the strong maximum principle.*

To state the concept of weak solution for these classes of linear boundary value problems we assume in addition that

$$\alpha_{ij} \in \mathscr{C}(\bar{\Omega}) \cap W^1_\infty(\Omega), \qquad 1 \leqslant i, j \leqslant N. \tag{2.5}$$

Then, a function φ is said to be a weak solution of

$$\begin{cases} \mathcal{L}\varphi = \lambda\varphi & \text{in} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

if $\varphi \in H_0^1(\Omega)$ and for each $\xi \in \mathscr{C}_c^\infty(\Omega)$

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial\varphi}{\partial x_i} \frac{\partial\xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i \xi \frac{\partial\varphi}{\partial x_i} + \int_\Omega \alpha_0 \xi\varphi = \lambda \int_\Omega \xi\varphi,$$

where

$$\tilde{\alpha}_i := \alpha_i + \sum_{j=1}^N \frac{\partial\alpha_{ij}}{\partial x_j} \in L_\infty(\Omega), \qquad 1 \leqslant i \leqslant N. \tag{2.6}$$

Hereafter, $\mathscr{C}_c^\infty(\Omega)$ stands for the space of $\mathscr{C}^\infty$ functions with compact support in Ω.

Now, let $n = (n_1, ..., n_N)$ denote the outward unit normal to Ω on Γ_1 and assume that the vector field $v = (v_1, ..., v_N)$ is given by

$$v_i := \sum_{j=1}^N \alpha_{ij} n_j, \qquad 1 \leqslant i \leqslant N; \tag{2.7}$$

in other words, ∂_v is the *conormal derivative*. Let $H^1_{\Gamma_0}(\Omega)$ denote the closure in $H^1(\Omega)$ of the set of functions $\mathscr{C}_c^\infty(\Omega \cup \Gamma_1)$. Then, a function φ is said to be a weak solution of (2.1) if $\varphi \in H^1_{\Gamma_0}(\Omega)$ and for each $\xi \in \mathscr{C}_c^\infty(\Omega \cup \Gamma_1)$

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial\varphi}{\partial x_i} \frac{\partial\xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i \xi \frac{\partial\varphi}{\partial x_i} + \int_\Omega \alpha_0 \xi\varphi = \lambda \int_\Omega \xi\varphi - \int_{\Gamma_1} b\xi\varphi .$$

3. MONOTONICITY PROPERTIES

In this section we are going to use the characterization of the strong maximum principle given by Theorem 2.1 to obtain some monotonicity properties of the principal eigenvalue. The following result shows the dominance of the principal eigenvalue of $\mathscr{L}$ under homogeneous Dirichlet boundary conditions.

Proposition 3.1. *Assume that $\Gamma_1 \neq \emptyset$. Then*

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < \sigma_1^\Omega[\mathscr{L}, \mathscr{D}].$$

Proof. Let $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}$ and $\varphi_{[\mathscr{L}, \mathscr{D}]}$ denote the principal eigenfunctions associated with the principal eigenvalues $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$ and $\sigma_1^\Omega[\mathscr{L}, \mathscr{D}]$, respectively. Since $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}$ is strongly positive, for each $x \in \Omega \cup \Gamma_1$ we have that $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}(x) > 0$ and hence, $\varphi_{[\mathscr{L}, \mathscr{B}(b)]} > 0$ on $\partial\Omega$. Thus, $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}$ provides us with a positive strict supersolution of $(\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)], \mathscr{D}, \Omega)$. Therefore, thanks to Theorem 2.1, we find that

$$0 < \sigma_1^\Omega[\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)], \mathscr{D}] = \sigma_1^\Omega[\mathscr{L}, \mathscr{D}] - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)].$$

This completes the proof. ∎

The following result shows the monotonicity of the principal eigenvalue with respect to the support domain.

Proposition 3.2. *Let Ω_0 be a proper subdomain of Ω of class $\mathscr{C}^2$ satisfying* (2.2). *Then,*

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)],$$

where $\mathscr{B}(b, \Omega_0)$ is the boundary operator defined by (2.3).

Proof. Let $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}$ denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$. Then,

$$\begin{cases} (\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)])\, \varphi_{[\mathscr{L}, \mathscr{B}(b)]} = 0 & \text{in } \Omega_0, \\ \varphi_{[\mathscr{L}, \mathscr{B}(b)]}(x) > 0 & \text{if } x \in \partial\Omega_0 \cap \Omega, \\ \varphi_{[\mathscr{L}, \mathscr{B}(b)]}(x) = 0 & \text{if } x \in \partial\Omega_0 \cap \Gamma_0, \\ \partial_\nu \varphi_{[\mathscr{L}, \mathscr{B}(b)]}(x) + b(x)\, \varphi_{[\mathscr{L}, \mathscr{B}(b)]}(x) = 0 & \text{if } x \in \partial\Omega_0 \cap \Gamma_1. \end{cases}$$

Moreover, $\partial\Omega_0 \cap \Omega \neq \emptyset$, since Ω_0 is a proper subdomain of Ω. Therefore, $\varphi_{[\mathscr{L}, \mathscr{B}(b)]}$ is a positive strict supersolution of $(\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)], \mathscr{B}(b, \Omega_0), \Omega_0)$ and it follows from Theorem 2.1 that

$$\begin{aligned} 0 &< \sigma_1^{\Omega_0}[\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)], \mathscr{B}(b, \Omega_0)] \\ &= \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)] - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]. \end{aligned}$$

This completes the proof. ∎

The following result shows the monotonicity of the principal eigenvalue with respect to the potential.

PROPOSITION 3.3. *Let $P_1, P_2 \in L_\infty(\Omega)$ such that $P_1 < P_2$ on a set of positive measure. Then*

$$\sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)] < \sigma_1^\Omega[\mathscr{L} + P_2, \mathscr{B}(b)].$$

Proof. Let φ_1 denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)]$. Then,

$$(\mathscr{L} + P_2 - \sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)])\, \varphi_1 = (P_2 - P_1)\, \varphi_1 > 0$$

in Ω. Thus, φ_1 is a positive strict supersolution of

$$(\mathscr{L} + P_2 - \sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)], \mathscr{B}(b), \Omega)$$

and therefore, thanks to Theorem 2.1,

$$\begin{aligned} 0 &< \sigma_1^\Omega[\mathscr{L} + P_2 - \sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)], \mathscr{B}(b)] \\ &= \sigma_1^\Omega[\mathscr{L} + P_2, \mathscr{B}(b)] - \sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)]. \end{aligned}$$

This completes the proof. ∎

As an immediate consequence, from this result we get the continuous dependence of the principal eigenvalue with respect to the potential.

COROLLARY 3.4. *Let $P_n \in L_\infty(\Omega)$, $n \geqslant 1$, be a sequence of potentials such that*

$$\lim_{n\to\infty} P_n = P \qquad \textit{in} \quad L_\infty(\Omega).$$

Then

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathscr{L} + P_n, \mathscr{B}(b)] = \sigma_1^\Omega[\mathscr{L} + P, \mathscr{B}(b)].$$

Proof. For any $\varepsilon > 0$ there exists a natural number $n(\varepsilon) \geqslant 1$ such that for each $n \geqslant n(\varepsilon)$

$$P - \varepsilon \leqslant P_n \leqslant P + \varepsilon \qquad \text{in} \quad \Omega.$$

Therefore, thanks to Proposition 3.3, for each $n \geqslant n(\varepsilon)$ we have that

$$\sigma_1^\Omega[\mathscr{L} + P, \mathscr{B}(b)] - \varepsilon \leqslant \sigma_1^\Omega[\mathscr{L} + P_n, \mathscr{B}(b)] \leqslant \sigma_1^\Omega[\mathscr{L} + P, \mathscr{B}(b)] + \varepsilon.$$

This completes the proof. ∎

The following result shows the monotonicity of the principal eigenvalue with respect to the weight function $b(x)$.

PROPOSITION 3.5. *Suppose $\Gamma_1 \neq \emptyset$ and let $b_1, b_2 \in \mathscr{C}(\Gamma_1)$ such that $b_1 \leqslant b_2$, $b_1 \neq b_2$. Then,*

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_2)].$$

Proof. Let φ_1 denote the principal eigenfunction associated with $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)]$.Then

$$(\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)])\, \varphi_1 = 0 \qquad \text{in} \qquad \Omega,$$

$\varphi_1 = 0$ on Γ_0, and

$$\partial_\nu \varphi_1 + b_2 \varphi_1 = (b_2 - b_1)\, \varphi_1 > 0$$

on Γ_1. Thus, φ_1 is a positive strict supersolution of $(\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)]$, $\mathscr{B}(b_2), \Omega)$. Therefore, thanks to Theorem 2.1,

$$0 < \sigma_1^\Omega[\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)], \mathscr{B}(b_2)] = \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_2)] - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)].$$

This completes the proof. ∎

COROLLARY 3.6. *Suppose $\Gamma_1 \neq \emptyset$, let $b_1, b_2 \in \mathscr{C}(\Gamma_1)$ such that $b_1 \leqslant b_2$, $b_1 \neq b_2$, and let $\Omega_0 \subset \Omega$ a subdomain of class $\mathscr{C}^2$ satisfying* (2.2). *Then*

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b_2, \Omega_0)]. \tag{3.1}$$

Proof. Assume that $\Omega = \Omega_0$. Then (3.1) becomes

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b_2)],$$

which is guaranteed by Proposition 3.5. It remains to show the result when Ω_0 is a proper subdomain of Ω. Thanks to Proposition 3.5,

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_2)].$$

Moreover, thanks to Proposition 3.2,

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_2)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b_2, \Omega_0)].$$

This completes the proof. ∎

Under the assumptions of Corollary 3.6, assume that $\partial\Omega_0 \cap \Gamma_1 = \emptyset$. Then, Ω_0 is a proper subdomain of Ω and $\mathscr{B}(b_2, \Omega_0) = \mathscr{D}$ on $\partial\Omega_0$. Hence, (3.1) becomes into

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{D}]. \tag{3.2}$$

This relation can be obtained directly from Proposition 3.1 and Proposition 3.2. Indeed, thanks to Proposition 3.1

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega}[\mathscr{L}, \mathscr{D}],$$

and hence, we find from Proposition 3.2 that

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b_1)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{D}].$$

4. POINT-WISE MIN-MAX CHARACTERIZATION

As a consequence from Theorem 2.1 the next point-wise min-max characterization of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ follows.

THEOREM 4.1. *Given $p > N$, let $\mathscr{P}_b$ denote the set of functions $\psi \in W_p^2(\Omega)$ such that $\psi(x) > 0$ for each $x \in \bar{\Omega}$ and $\mathscr{B}(b)\,\psi > 0$ on $\partial\Omega$. Then*

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)] = \sup_{\psi \in \mathscr{P}_b} \inf_{x \in \Omega} \frac{\mathscr{L}\psi}{\psi}. \tag{4.1}$$

Proof. Fix $\lambda < \sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$. Then

$$\sigma_1^{\Omega}[\mathscr{L} - \lambda, \mathscr{B}(b)] > 0$$

and hence, thanks to Theorem 2.1, $(\mathscr{L}-\lambda, \mathscr{B}(b), \Omega)$ satisfies the strong maximum principle. Thus, the unique solution ψ_1 of the problem

$$\begin{cases} (\mathscr{L}-\lambda)\,\psi_1 = 1 & \text{in} \quad \Omega, \\ \psi_1 = 1 & \text{on} \quad \Gamma_0, \\ \partial_\nu \psi_1 + b\psi_1 = 1 & \text{on} \quad \Gamma_1, \end{cases}$$

is strongly positive. Moreover, $\psi_1 \in W_p^2(\Omega)$. In particular, $\psi_1 \in \mathscr{P}_b$ and $\mathscr{P}_b \neq \emptyset$. We have that $\psi_1(x) > 0$ for each $x \in \bar{\Omega}$. Hence,

$$\lambda < \frac{\mathscr{L}\psi_1}{\psi_1} \qquad \text{in} \quad \Omega.$$

Thus,

$$\lambda \leqslant \inf_{x \in \Omega} \frac{\mathscr{L}\psi_1}{\psi_1} \leqslant \sup_{\psi \in \mathscr{P}_b} \inf_{x \in \Omega} \frac{\mathscr{L}\psi}{\psi}. \tag{4.2}$$

Since (4.2) is valid for any $\lambda < \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]$, we find that

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] \leqslant \sup_{\psi \in \mathscr{P}_b} \inf_{x \in \Omega} \frac{\mathscr{L}\psi}{\psi}.$$

To complete the proof of (4.1) we argue by contradiction assuming that

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < \sup_{\psi \in \mathscr{P}_b} \inf_{x \in \Omega} \frac{\mathscr{L}\psi}{\psi}.$$

Then there exists $\varepsilon > 0$ and $\psi \in \mathscr{P}_b$ such that for each $x \in \Omega$

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] + \varepsilon < \frac{\mathscr{L}\psi(x)}{\psi(x)}.$$

Thus,

$$\begin{cases} (\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] - \varepsilon)\,\psi > 0 & \text{in} \quad \Omega, \\ \mathscr{B}(b)\,\psi > 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and hence, ψ is a positive strict supersolution of $(\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] - \varepsilon, \mathscr{B}(b), \Omega)$. Therefore, thanks to Theorem 2.1

$$0 < \sigma_1^\Omega[\mathscr{L} - \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] - \varepsilon, \mathscr{B}(b)] = -\varepsilon,$$

which is impossible. This contradiction completes the proof. ∎

5. CONCAVITY WITH RESPECT TO THE POTENTIAL

In this section we show the concavity of the map

$$L_\infty(\Omega) \longmapsto \mathbf{R}$$
$$P \to \sigma_1^\Omega[\mathscr{L}+P, \mathscr{B}(b)]$$

with respect to P. Some previous less general results of this type were given in [21, 25, 29]. The proof given here follows the same scheme as the proof of Theorem 3.3 in [29], and it uses a device coming from [9].

THEOREM 5.1. *For each $P_1,\ P_2 \in L_\infty(\Omega)$ and $t \in [0,1]$ the following inequality holds*

$$\sigma_1^\Omega[\mathscr{L}+tP_1+(1-t)\,P_2, \mathscr{B}(b)] \geqslant t\sigma_1^\Omega[\mathscr{L}+P_1, \mathscr{B}(b)] + (1-t)\,\sigma_1^\Omega[\mathscr{L}+P_2, \mathscr{B}(b)]. \qquad (5.1)$$

Proof. Since $\mathscr{L}$ is strongly uniformly elliptic in Ω, for any $x \in \bar{\Omega}$ the bilinear form

$$\langle a, b\rangle := \sum_{i,j=1}^N \alpha_{ij}(x)\, a_i b_j,$$

where $a=(a_1, ..., a_N)$, $b=(b_1, ..., b_N) \in \mathbf{R}^N$, defines an scalar product in $\mathbf{R}^N$ and hence, thanks to Hölder inequality we have that

$$2\sum_{i,j=1}^N \alpha_{ij}(x)\, a_i b_j \leqslant \sum_{i,j=1}^N \alpha_{ij}(x)\, a_i a_j + \sum_{i,j=1}^N \alpha_{ij}(x)\, b_i b_j.$$

From this inequality it is easily seen that for each $p>N$ the mapping $G: W_p^2(\Omega) \to L_p(\Omega)$ defined by

$$G(u) := (\mathscr{L}-\alpha_0)\,u+\alpha_0 - \sum_{i,j=1}^N \alpha_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}, \qquad u \in W_p^2(\Omega),$$

is concave; that is, for each $u_1,\ u_2 \in W_p^2(\Omega)$ and $t \in [0,1]$ the following inequality is satisfied

$$G(tu_1+(1-t)\,u_2) \geqslant tG(u_1)+(1-t)\,G(u_2).$$

Note that for any $\psi \in \mathscr{P}_b$ the following relation holds

$$\frac{\mathscr{L}\psi}{\psi} = (\mathscr{L}-\alpha_0)\,\theta+\alpha_0 - \sum_{i,j=1}^N \alpha_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} = G(\theta), \qquad \theta := \log\psi.$$

Consider $P_1, P_2 \in L_\infty(\Omega)$, $t \in [0,1]$ and $\psi_1, \psi_2 \in \mathscr{P}_b$ arbitrary. Set

$$\theta_i := \log \psi_i, \qquad i = 1, 2 .$$

Taking into account that whenever $\psi \in \mathscr{P}_b$ one has that $\psi, \frac{1}{\psi} \in L_\infty(\Omega)$ and $\nabla\psi \in L_\infty(\Omega, \mathbf{R}^N)$, it is easily seen that $\psi_1^t \psi_2^{1-t} \in \mathscr{P}_b$. Then

$$\begin{aligned}
&\frac{[\mathscr{L} + t\, P_1 + (1-t)\, P_2](\psi_1^t \psi_2^{1-t})}{\psi_1^t \psi_2^{1-t}} \\
&\quad = t\, P_1 + (1-t)\, P_2 + \frac{\mathscr{L}\psi_1^t \psi_2^{1-t}}{\psi_1^t \psi_2^{1-t}} \\
&\quad = t\, P_1 + (1-t)\, P_2 + G(\log(\psi_1^t \psi_2^{1-t})) \\
&\quad = t\, P_1 + (1-t)\, P_2 + G(t \log \psi_1 + (1-t) \log \psi_2) \\
&\quad \geqslant t\, P_1 + (1-t)\, P_2 + t\, G(\theta_1) + (1-t)\, G(\theta_2) \\
&\quad = t\, \frac{(\mathscr{L} + P_1)\, \psi_1}{\psi_1} + (1-t)\, \frac{(\mathscr{L} + P_2)\, \psi_2}{\psi_2} \\
&\quad \geqslant t \inf_\Omega \frac{(\mathscr{L} + P_1)\, \psi_1}{\psi_1} + (1-t) \inf_\Omega \frac{(\mathscr{L} + P_2)\, \psi_2}{\psi_2}.
\end{aligned}$$

Thus, thanks to Theorem 4.1 we find that

$$\begin{aligned}
&\sigma_1^\Omega[\mathscr{L} + t\, P_1 + (1-t)\, P_2, \mathscr{B}(b)] \\
&\qquad \geqslant t \inf_\Omega \frac{(\mathscr{L} + P_1)\, \psi_1}{\psi_1} + (1-t) \inf_\Omega \frac{(\mathscr{L} + P_2)\, \psi_2}{\psi_2}.
\end{aligned}$$

Since this inequality is satisfied for all $\psi_1, \psi_2 \in \mathscr{P}_b$, taking supremums on its right hand side with respect to ψ_1 and ψ_2 gives

$$\begin{aligned}
&\sigma_1^\Omega[\mathscr{L} + t\, P_1 + (1-t)\, P_2, \mathscr{B}(b)] \\
&\qquad \geqslant t\, \sigma_1^\Omega[\mathscr{L} + P_1, \mathscr{B}(b)] + (1-t)\, \sigma_1^\Omega[\mathscr{L} + P_2, \mathscr{B}(b)].
\end{aligned}$$

This concludes the proof. ∎

6. THE CONCEPT OF STABILITY

The concept of stability of a domain goes back to Babuška [7] and Babuška and Vyborny [8] where it was used to generalize some pioneer results of Courant and Hilbert [11] on the continuous variation with respect to the domain Ω of the eigenvalues of a selfadjoint differential operator $\mathscr{L}$ under homogeneous Dirichlet boundary conditions on $\partial\Omega$. Later, this concept was shown to play a central rol in potential theory, since it provides us with all domains for which the Dirichlet problem makes sense (cf. [1] and the references therein). Moreover, it was required in [29] to show the continuous dependence of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{D}]$ with respect to Ω for a general class of differential operators $\mathscr{L}$ not necessarily selfadjoint.

Nevertheless, when dealing with Neumann boundary conditions Courant and Hilbert [11] observed that the continuous dependence of the principal eigenvalue with respect to the domain may fail and hence, the continuous dependence of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ with respect to Ω is not necessarily true. In fact, it fails if $\Gamma_0 = \emptyset$, $b = 0$, $\mathscr{L} = -\Delta$ and $v = n$ is the outward unit normal. Thus, a rather natural question arises: Under our general boundary conditions, is there some general class of perturbations of the domain for which the continuous dependence of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ on Ω holds true? In other words, how should the concept of stability of the open set Ω be extended so that we can get the continuous dependence of the principal eigenvalue?

In Section 7 we shall show that if $\Gamma_0 \neq \emptyset$, ∂_v is the conormal derivative associated to $\mathscr{L}$ and Ω perturbs in such a way that the *Neumann boundary* Γ_1 is kept fixed, then $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ varies continuously with Ω. Therefore, we adopt the following concepts.

Definition 6.1. Let Ω_0 be a bounded domain of $\mathbf{R}^N$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$, and Ω_n, $n \geqslant 1$, a sequence of bounded domains of $\mathbf{R}^N$ with boundary $\partial\Omega_n = \Gamma_0^n \cup \Gamma_1$ of class $\mathscr{C}^2$ where Γ_0^n and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_n$, $n \geqslant 1$. Then:

(a) It is said that Ω_n converges to Ω_0 from the exterior if for each $n \geqslant 1$

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} \bar{\Omega}_n = \bar{\Omega}_0 .$$

(b) It is said that Ω_n converges to Ω_0 from the interior if for each $n \geqslant 1$

$$\Omega_n \subset \Omega_{n+1} \subset \Omega_0 \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} \Omega_n = \Omega_0 .$$

(c) It is said that Ω_n converges to Ω_0 if there exist two sequences of bounded smooth domains, say Ω_n^I and Ω_n^E, $n \geqslant 1$, such that Ω_n^I converges to Ω_0 from the interior, Ω_n^E converges to Ω_0 from the exterior, and for each $n \geqslant 1$

$$\Omega_n^I \subset \Omega_0 \cap \Omega_n, \qquad \Omega_0 \cup \Omega_n \subset \Omega_n^E.$$

Remark 6.2. It should be pointed out that if Ω_n is a sequence of bounded domains converging to Ω_0 from the exterior in the sense of Definition 6.1a), then

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega_{n+1}) = \operatorname{dist}(\Gamma_1, \Gamma_0^0) > 0$$

and

$$\operatorname{dist}(\Gamma_1, \partial\Omega_{n+1} \cap \Omega_n) = \operatorname{dist}(\Gamma_1, \Gamma_0^{n+1}) > 0.$$

In the same way, if Ω_n is a sequence of bounded domains converging to Ω_0 from the interior in the sense of Definition 6.1b), then

$$\operatorname{dist}(\Gamma_1, \partial\Omega_n \cap \Omega_{n+1}) = \operatorname{dist}(\Gamma_1, \Gamma_0^n) > 0$$

and

$$\operatorname{dist}(\Gamma_1, \partial\Omega_{n+1} \cap \Omega_0) = \operatorname{dist}(\Gamma_1, \Gamma_0^{n+1}) > 0.$$

DEFINITION 6.3. Let Ω_0 be a bounded domain of $\mathbf{R}^N$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$, where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$. It is said that Ω_0 is stable if for any sequence of bounded smooth domains Ω_n, $n \geqslant 1$, converging to Ω_0 from the exterior in the sense of Definition 6.1-a) the following relation holds

$$\bigcap_{n=1}^{\infty} H^1_{\Gamma_0^n}(\Omega_n) = H^1_{\Gamma_0^0}(\Omega_0),$$

where the functions of $H^1_{\Gamma_0^0}(\Omega_0)$ are regarded as functions of $H^1_{\Gamma_0^n}(\Omega_n)$ by extending them by zero outside Ω_0. We recall that $H^1_{\Gamma_0^n}(\Omega_n)$ stands for the closure in $H^1(\Omega_n)$ of the set of functions $\mathscr{C}_c^\infty(\Omega_n \cup \Gamma_1)$, $n \geqslant 0$.

The following result shows that if the boundary of Ω_0 is $\mathscr{C}^1$, then Ω_0 is stable in the sense of Definition 6.2. It is one of the pivotal results from which we will obtain our main theorem on the continuous dependence of the principal eigenvalue with respect to Ω_0.

THEOREM 6.4. *Let Ω_0 be a bounded domain of $\mathbf{R}^N$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ of class $\mathscr{C}^1$, where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$. Then, Ω_0 is stable.*

To prove this result we need the following very sharp version of Theorem 3.7 in [39].

THEOREM 6.5. *Let Ω be a bounded domain of $\mathbf{R}^N$ of class $\mathscr{C}^1$ with boundary*

$$\partial\Omega = \Gamma_0 \cup \Gamma_1,$$

where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$ and consider any proper subdomain $\Omega_0 \subset \Omega$ of class $\mathscr{C}^1$ with boundary

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$. Let $H^1_{\Gamma_0^0}(\Omega_0)$ denote the closure in $H^1(\Omega_0)$ of the set of functions $\mathscr{C}_c^\infty(\Omega_0 \cup \Gamma_1)$. Then

$$H^1_{\Gamma_0^0}(\Omega_0) = \{u \in H^1(\Omega) : \operatorname{supp} u \subset \bar{\Omega}_0\}.$$

In the proof of this result we are going to use the following well known concept.

DEFINITION 6.6. It is said that Ω_0 satisfies the segment property if for any $x \in \partial\Omega_0$ there exist a neighborhood U_x of x and a vector $v_x \in \mathbf{R}^N \setminus \{0\}$ such that for each $t \in (0, 1)$

$$U_x \cap \bar{\Omega}_0 + tv_x \subset \Omega_0.$$

Remark 6.7. If Ω_0 is of class $\mathscr{C}^1$, then it satisfies the segment property.

Proof of Theorem 6.4. Let $u \in H^1(\Omega)$ be such that

$$\operatorname{supp} u \subset \bar{\Omega}_0.$$

Since $\Gamma_1 \subset \partial\Omega \cap \partial\Omega_0$ and Ω_0 is a subdomain of Ω, there exists $\varepsilon > 0$ such that

$$U^\varepsilon \cap \Omega \subset \Omega_0, \qquad U^\varepsilon := \Gamma_1 + B_\varepsilon,$$

where B_ε is the ball of radius ε centered at the origin. Moreover, since $\Gamma_0^0 \cap \Gamma_1 = \emptyset$, ε can be chosen sufficiently small so that

$$U^\varepsilon \cap \Gamma_0^0 = \emptyset. \tag{6.1}$$

Let $\eta \in \mathscr{C}_c^\infty(U^\varepsilon)$ such that

$$\eta(x) = 1 \qquad \text{for each} \quad x \in U^{\frac{\varepsilon}{2}}.$$

Since $u \in H^1(\Omega)$, $u \in H^1(\Omega_0)$. Hence, due to the fact that the set of restrictions

$$\{\psi|_{\Omega_0} : \psi \in \mathscr{C}_c^\infty(\mathbf{R}^N)\}$$

is dense in $H^1(\Omega_0)$, there exists a sequence $\psi_n \in \mathscr{C}_c^\infty(\mathbf{R}^N)$, $n \geqslant 1$, such that

$$\lim_{n\to\infty} \|\psi_n|_{\Omega_0} - u\|_{H^1(\Omega_0)} = 0. \tag{6.2}$$

Consider the new functions

$$\xi_n := (\eta\psi_n)|_{\Omega_0}, \qquad n \geqslant 1.$$

Thanks to (6.1), for each $n \geqslant 1$ we have that

$$\xi_n \in \mathscr{C}_c^\infty(\Omega_0 \cup \Gamma_1).$$

Moreover, thanks to (6.2),

$$\lim_{n\to\infty} \|\xi_n - \eta u\|_{H^1(\Omega_0)} = 0. \tag{6.3}$$

By Remark 6.6, Ω_0 satisfies the segment property. Hence, for each $x \in \Gamma_0^0$ there exist a neighborhood U_x of x and a vector $v_x \in \mathbf{R}^N \setminus \{0\}$ such that for each $t \in (0, 1)$

$$U_x \cap \bar{\Omega}_0 + tv_x \subset \Omega_0. \tag{6.4}$$

By the compactness of Γ_0^0 there exist a natural number $m \geqslant 1$ and m points $x_j \in \Gamma_0^0$, $1 \leqslant j \leqslant m$, such that

$$\Gamma_0^0 \subset \bigcup_{j=1}^m U_{x_j}.$$

Let U_{m+1} be an open set such that $\bar{U}_{m+1} \subset \Omega_0$ and the system

$$\{U_{x_1}, \ldots, U_{x_m}, U_{m+1}\}$$

provides us with a covering of $\bar{\Omega}_0 \setminus U^{\varepsilon/2}$, and let

$$\{\beta_1, ..., \beta_m, \beta_{m+1}\}$$

be a partition of the unity subordinated to that covering. Set

$$\gamma_j := \beta_j(1-\eta)\, u, \qquad 1 \leqslant j \leqslant m+1,$$

and consider the translations

$$\gamma_j^t := \gamma_j(\cdot - tv_{x_j}), \qquad 1 \leqslant j \leqslant m, \quad 0 < t < 1.$$

By construction,

$$\operatorname{supp} \gamma_{m+1} \subset \operatorname{supp} \beta_{m+1} \subset U_{m+1}$$

and $\bar{U}_{m+1} \subset \Omega_0$. Hence,

$$\gamma_{m+1} \in H_0^1(\Omega_0)$$

and therefore, there exists a sequence $\xi_n^{m+1} \in \mathscr{C}_c^\infty(\Omega_0)$, $n \geqslant 1$, such that

$$\lim_{n \to \infty} \|\xi_n^{m+1} - \gamma_{m+1}\|_{H^1(\Omega_0)} = 0. \tag{6.5}$$

Moreover, since we are assuming that $\operatorname{supp} u \subset \bar{\Omega}_0$, for each $1 \leqslant j \leqslant m$ we find that

$$\operatorname{supp} \beta_j \subset U_{x_j} \cap \operatorname{supp} u \subset U_{x_j} \cap \bar{\Omega}_0 .$$

Thus, thanks to (6.4),

$$\operatorname{supp} \gamma_j^t \subset U_{x_j} \cap \bar{\Omega}_0 + tv_{x_j} \subset \Omega_0, \qquad 1 \leqslant j \leqslant m, \quad 0 < t < 1,$$

and hence,

$$\gamma_j^t \in H_0^1(\Omega_0), \qquad 1 \leqslant j \leqslant m, \quad 0 < t < 1. \tag{6.6}$$

By the continuity of the translation operator, for each natural number $n \geqslant 1$ there exists $t_n \in (0, 1)$ such that

$$\|\gamma_j^{t_n} - \gamma_j\|_{H^1(\Omega_0)} \leqslant \frac{1}{n}, \qquad 1 \leqslant j \leqslant m. \tag{6.7}$$

Moreover, thanks to (6.6), for each $n \geqslant 1$ and $1 \leqslant j \leqslant m$ there exists

$$\xi_n^j \in \mathscr{C}_c^\infty(\Omega_0)$$

such that

$$\|\gamma_j^{t_n} - \xi_n^j\|_{H^1(\Omega_0)} \leqslant \frac{1}{n}. \tag{6.8}$$

Therefore, thanks to (6.7) and (6.8), we find that

$$\lim_{n\to\infty} \|\xi_n^j - \gamma_j\|_{H^1(\Omega_0)} = 0, \qquad 1 \leqslant j \leqslant m. \tag{6.9}$$

Now, consider the sequence

$$\varphi_n := \xi_n + \sum_{j=1}^{m+1} \xi_n^j, \qquad n \geqslant 1.$$

By construction, for each $n \geqslant 1$ we have that

$$\varphi_n \in \mathscr{C}_c^\infty(\Omega_0 \cup \Gamma_1).$$

Moreover, thanks to (6.3), (6.5), and (6.9),

$$\lim_{n\to\infty} \varphi_n = \eta u + \sum_{j=1}^{m+1} \beta_j (1-\eta)\, u \qquad \text{in} \quad H^1(\Omega_0).$$

In $\bar{\Omega}_0 \setminus U^{\varepsilon/2}$ we have that

$$\sum_{j=1}^{m+1} \beta_j = 1,$$

while $\eta = 1$ in $U^{\varepsilon/2}$. Thus,

$$\lim_{n\to\infty} \varphi_n = u \qquad \text{in} \quad H^1(\Omega_0),$$

and therefore,

$$u \in H^1_{\Gamma_0^0}(\Omega_0).$$

To show the other inclusion consider $u \in H^1_{\Gamma_0^0}(\Omega_0)$. By definition, there exists a sequence

$$\varphi_n \in \mathscr{C}_c^\infty(\Omega_0 \cup \Gamma_1), \qquad n \geqslant 1, \tag{6.10}$$

such that

$$\lim_{n\to\infty} \|\varphi_n - u\|_{H^1(\Omega_0)} = 0. \tag{6.11}$$

Now, consider the new sequence

$$\psi_n := \begin{cases} \varphi_n & \text{in} \quad \Omega_0 \cup \Gamma_1, \\ 0 & \text{in} \quad \bar{\Omega} \setminus (\Omega_0 \cup \Gamma_1), \end{cases} \qquad n \geqslant 1.$$

Thanks to (6.10), (6.11), and due to the fact that $\Gamma_0^0 \cap \Gamma_1 = \emptyset$, we have that

$$\psi_n \in \mathscr{C}_c^\infty(\Omega \cup \Gamma_1), \qquad n \geqslant 1.$$

Moreover, ψ_n, $n \geqslant 1$, is a Cauchy sequence in $H^1(\Omega)$. Thus, there exists $\psi \in H^1(\Omega)$ such that

$$\lim_{n \to \infty} \|\psi_n - \psi\|_{H^1(\Omega)} = 0. \tag{6.12}$$

In particular,

$$\lim_{n \to \infty} \psi_n = \psi \qquad \text{almost everywhere in} \quad \Omega,$$

and hence $\psi = 0$ in $\Omega \setminus \Omega_0$, since $\psi_n = 0$ in $\Omega \setminus \Omega_0$ for each $n \geqslant 1$. Thus,

$$\operatorname{supp} \psi \subset \bar{\Omega}_0.$$

On the other hand, it follows from (6.12) that

$$\lim_{n \to \infty} \|\psi_n|_{\Omega_0} - \psi|_{\Omega_0}\|_{H^1(\Omega_0)} = 0. \tag{6.13}$$

Moreover, by definition,

$$\varphi_n = \psi_n|_{\Omega_0}, \qquad n \geqslant 1.$$

Thus, we find from (6.11) and (6.13) that

$$u = \psi|_{\Omega_0}.$$

Therefore, $u \in H^1(\Omega)$ and

$$\operatorname{supp} u \subset \bar{\Omega}_0.$$

This completes the proof. ∎

We are now ready to prove Theorem 6.3.

Proof of Theorem 6.3. Let Ω_n, $n \geqslant 1$, be a sequence of bounded smooth domains converging to Ω_0 from the exterior in the sense of Definition 6.1(a). We have to show that

$$\bigcap_{n=1}^{\infty} H^1_{\Gamma_0^n}(\Omega_n) = H^1_{\Gamma_0^0}(\Omega_0). \tag{6.14}$$

If $\Omega_1 = \Omega_0$, then $\Omega_n = \Omega_0$ for each $n \geqslant 1$ and therefore, (6.14) holds true. So, for the rest of the proof we shall assume that Ω_0 is a proper subdomain of Ω_1.

Assume

$$u \in \bigcap_{n=1}^{\infty} H^1_{\Gamma_0^n}(\Omega_n).$$

Then from the definition it is easily seen that

$$u \in H^1(\Omega_n), \qquad \operatorname{supp} u \subset \bar{\Omega}_n, \qquad n \geqslant 1,$$

and hence,

$$u \in H^1(\Omega_1), \qquad \operatorname{supp} u \subset \bigcap_{n=1}^{\infty} \bar{\Omega}_n = \bar{\Omega}_0.$$

Therefore, thanks to Theorem 6.4,

$$u \in H^1_{\Gamma_0^0}(\Omega_0). \tag{6.15}$$

Now, assume (6.15) and fix $n \geqslant 1$. If $\Omega_0 = \Omega_n$, then

$$u \in H^1_{\Gamma_0^n}(\Omega_n),$$

while if Ω_0 is a proper subdomain of Ω_n the extension of u by zero outside Ω_0, say $\tilde{u}$, satisfies

$$\tilde{u} \in H^1_{\Gamma_0^n}(\Omega_n).$$

This completes the proof. ∎

7. CONTINUOUS DEPENDENCE WITH RESPECT TO Ω

In this section we analyze the continuous dependence of $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ with respect to perturbations of the domain Ω around its Dirichlet boundary Γ_0 in the special case when ∂_ν is the conormal derivative with respect to

$\mathscr{L}$, i.e., when condition (2.7) is satisfied. So, for the remainder of this section condition (2.7) will be assumed, although we believe that this assumption is needed exclusively by technical reasons.

Let $\mu > 0$ denote the ellipticity constant of $\mathscr{L}$. Then, thanks to (2.7), we have

$$\langle v, n \rangle = \sum_{i,j=1}^{N} \alpha_{ij} n_j n_i \geqslant \mu\, |n|^2 = \mu > 0,$$

and therefore, v is an outward pointing nowhere tangent vector field. Also, note that if $\alpha_{ij} \in \mathscr{C}^1(\bar{\Omega})$, $1 \leqslant i, j \leqslant N$, then $v \in \mathscr{C}^1(\Gamma_1, \mathbf{R}^N)$, since Γ_1 is of class $\mathscr{C}^2$.

The continuous dependence of the principal eigenvalue with respect to the domain is based upon the following result, which provides us with the *continuous dependence from the exterior*.

THEOREM 7.1. *Assume that*

$$\alpha_{ij} \in \mathscr{C}^1(\bar{\Omega}), \qquad \alpha_i \in \mathscr{C}(\bar{\Omega}), \qquad 1 \leqslant i, j \leqslant N, \tag{7.1}$$

and that condition (2.7) *is satisfied.*

Let Ω_0 be a proper subdomain of Ω with boundary of class $\mathscr{C}^2$ such that

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$ and let Ω_n, $n \geqslant 1$, be a sequence of bounded domains of $\mathbf{R}^N$ of class $\mathscr{C}^2$ converging to Ω_0 from the exterior and such that $\Omega_n \subset \Omega$, $n \geqslant 1$. For each $n \geqslant 0$, let $\mathscr{B}_n(b)$ denote the boundary operator defined by

$$\mathscr{B}_n(b)\, u := \begin{cases} u & \text{on } \Gamma_0^n, \\ \partial_v u + bu & \text{on } \Gamma_1, \end{cases}$$

where

$$\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1, \qquad n \geqslant 0,$$

and denote by $(\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \varphi_n)$ the principal eigen-pair associated with $(\mathscr{L}, \mathscr{B}_n(b), \Omega_n)$, where the principal eigenfunction is assumed to be normalized so that

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1, \qquad n \geqslant 0.$$

Then $\varphi_0 \in W^2_{\mathscr{B}_0(b)}(\Omega_0)$ *and*

$$\lim_{n\to\infty} \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)], \qquad \lim_{n\to\infty} \|\varphi_n|_{\Omega_0} - \varphi_0\|_{H^1(\Omega_0)} = 0.$$

Proof. The existence and the uniqueness of the principal eigen-pairs

$$(\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \varphi_n), \qquad n \geqslant 0,$$

is guaranteed by Theorem 12.1 of [3]. By construction, for each $n \geqslant 1$

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n \subset \Omega,$$

and hence, thanks to Remark 6.2 and Proposition 3.2 we find that

$$\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] \leqslant \sigma_1^{\Omega_{n+1}}[\mathscr{L}, \mathscr{B}_{n+1}(b)] \leqslant \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)], \qquad n \geqslant 1.$$

Therefore, the limit

$$\sigma_1^E := \lim_{n\to\infty} \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] \tag{7.2}$$

is well defined. We have to show that

$$\sigma_1^E = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)]. \tag{7.3}$$

Thanks to Theorem 12.1 of [3],

$$\varphi_n \in W^2_{\mathscr{B}_n(b)}(\Omega_n) \subset H^2(\Omega_n), \qquad n \geqslant 0.$$

Now, set

$$\tilde{\varphi}_n := \begin{cases} \varphi_n & \text{in} \quad \Omega_n, \\ 0 & \text{in} \quad \Omega \backslash \Omega_n, \end{cases} \qquad n \geqslant 0.$$

Since $\varphi_n \in H^1(\Omega_n)$ and $\varphi_n = 0$ on Γ_0^n, for each $n \geqslant 0$ we have $\tilde{\varphi}_n \in H^1(\Omega)$. Moreover,

$$\|\tilde{\varphi}_n\|_{H^1(\Omega)} = \|\varphi_n\|_{H^1(\Omega_n)} = 1, \qquad n \geqslant 0. \tag{7.4}$$

Thus, since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exists a subsequence of $\tilde{\varphi}_n$, $n \geqslant 1$, relabeled by n, such that

$$\lim_{n\to\infty} \|\tilde{\varphi}_n - \tilde{\varphi}\|_{L^2(\Omega)} = 0 \tag{7.5}$$

for some function $\tilde{\varphi} \in L^2(\Omega)$. In particular,

$$\lim_{n \to \infty} \tilde{\varphi}_n(x) = \tilde{\varphi}(x) \qquad \text{a.e. in} \quad \Omega. \tag{7.6}$$

We claim that

$$\operatorname{supp} \tilde{\varphi} \subset \bar{\Omega}_0 . \tag{7.7}$$

Indeed, pick

$$x \notin \bar{\Omega}_0 = \bigcap_{n=1}^{\infty} \bar{\Omega}_n .$$

Then, since $\bar{\Omega}_n$, $n \geqslant 1$, is a decreasing sequence of compact sets, there exists a natural number $n_0 \geqslant 1$ such that $x \notin \bar{\Omega}_n$ for each $n \geqslant n_0$. Hence,

$$\tilde{\varphi}_n(x) = 0, \qquad n \geqslant n_0 .$$

Thus,

$$\lim_{n \to \infty} \tilde{\varphi}_n(x) = 0 \qquad \text{if} \quad x \notin \bar{\Omega}_0 .$$

By the uniqueness of the limit in (7.6) we have that

$$\tilde{\varphi} = 0 \qquad \text{in} \quad \Omega \backslash \bar{\Omega}_0 ,$$

and therefore (7.7) is shown. Note that $\varphi_n(x) > 0$ for each $x \in \Omega_n \cup \Gamma_1$ and $n \geqslant 0$, since φ_n is strongly positive in Ω_n. In particular, $\varphi_n(x) > 0$ for each $x \in \Omega_0 \cup \Gamma_1$ and $n \geqslant 0$. Hence, (7.6) implies

$$\tilde{\varphi} \geqslant 0 \qquad \text{in} \quad \Omega_0 . \tag{7.8}$$

We now analyze the limiting behavior of the traces of φ_n, $n \geqslant 1$, on Γ_1. By our regularity requirements on $\partial\Omega_0$, it follows from the trace theorem (e.g., Theorem 8.7 of [39]) that there exists a linear continuous operator

$$\gamma_1 := H^1(\Omega_0) \to W_2^{\frac{1}{2}}(\Gamma_1)$$

such that

$$\gamma_1 u = u|_{\Gamma_1} \qquad \text{for each} \quad u \in H^1(\Omega_0). \tag{7.9}$$

Such an operator is called the trace operator on Γ_1.

For each $n \geqslant 1$, let i_n denote the canonical injection

$$i_n: H^1(\Omega_n) \to H^1(\Omega_0);$$

i_n is the restriction to Ω_0 of the functions of $H^1(\Omega_n)$. Note that for each $n \geqslant 1$

$$\|i_n\|_{\mathcal{L}(H^1(\Omega_n), H^1(\Omega_0))} \leqslant 1. \tag{7.10}$$

Now, setting

$$T_n := \gamma_1 \circ i_n, \qquad n \geqslant 1,$$

we find from (7.10) that

$$\|T_n\|_{\mathcal{L}(H^1(\Omega_n), W_2^{1/2}(\Gamma_1))} \leqslant \|\gamma_1\|_{\mathcal{L}(H^1(\Omega_0), W_2^{1/2}(\Gamma_1))}, \qquad n \geqslant 1.$$

In particular, these operators are uniformly bounded. Moreover, for each $n \geqslant 1$ we have that

$$\varphi_n|_{\Gamma_1} = i_n(\varphi_n)|_{\Gamma_1} = T_n \varphi_n \in W_2^{1/2}(\Gamma_1),$$

and hence

$$\|\varphi_n|_{\Gamma_1}\|_{W_2^{1/2}(\Gamma_1)} = \|T_n \varphi_n\|_{W_2^{1/2}(\Gamma_1)} \leqslant \|\gamma_1\|_{\mathcal{L}(H^1(\Omega_0), W_2^{1/2}(\Gamma_1))}, \qquad n \geqslant 1,$$

where we have used the normalization condition.

On the other hand, the embedding $W_2^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L_2(\Gamma_1)$ is compact, since Γ_1 is compact (e.g. Theorem 7.10 of [39]), and therefore there exists a subsequence of φ_n, $n \geqslant 1$, again labeled by n, such that

$$\lim_{n \to \infty} \|\varphi_n|_{\Gamma_1} - \varphi^*\|_{L_2(\Gamma_1)} = 0 \tag{7.11}$$

for some $\varphi^* \in L_2(\Gamma_1)$.

We now show that the corresponding sequence $\tilde{\varphi}_n$, $n \geqslant 1$, is a Cauchy sequence in $H^1(\Omega)$. Thanks to (7.5) this shows that

$$\lim_{n \to \infty} \|\tilde{\varphi}_n - \tilde{\varphi}\|_{H^1(\Omega)} = 0. \tag{7.12}$$

Indeed, suppose that k and m are natural numbers such that $1 \leqslant k \leqslant m$. Then, $\Omega_m \subset \Omega_k$ and due to the fact that $\mathcal{L}$ is strongly uniformly elliptic in

$\bar{\Omega}$ integrating by parts and taking into account that $\varphi_n = 0$ on Γ_0^n for each $n \geqslant 1$ gives

$$\begin{aligned}
&\mu \,\|\nabla(\tilde{\varphi}_k - \tilde{\varphi}_m)\|^2_{L_2(\Omega)} \\
&\quad \leqslant \sum_{i,j=1}^{N} \int_\Omega \alpha_{ij} \frac{\partial}{\partial x_i} (\tilde{\varphi}_k - \tilde{\varphi}_m) \frac{\partial}{\partial x_j} (\tilde{\varphi}_k - \tilde{\varphi}_m) \\
&\quad = \sum_{i,j=1}^{N} \left\{ \int_{\Omega_k} \alpha_{ij} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} + \int_{\Omega_m} \alpha_{ij} \frac{\partial \varphi_m}{\partial x_i} \frac{\partial \varphi_m}{\partial x_j} - 2 \int_{\Omega_m} \alpha_{ij} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_m}{\partial x_j} \right\} \\
&\quad = -\sum_{i,j=1}^{N} \left\{ \int_{\Omega_k} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial \varphi_k}{\partial x_i} \right) \varphi_k + \int_{\Omega_m} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial \varphi_m}{\partial x_i} \right) \varphi_m \right. \\
&\qquad \left. -2 \int_{\Omega_m} \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial \varphi_k}{\partial x_i} \right) \varphi_m \right\} \\
&\qquad + \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \left\{ \frac{\partial \varphi_k}{\partial x_i} \varphi_k n_j + \frac{\partial \varphi_m}{\partial x_i} \varphi_m n_j - 2 \frac{\partial \varphi_k}{\partial x_i} \varphi_m n_j \right\}.
\end{aligned}$$

From this relation, thanks to the fact that φ_n is the principal eigenfunction associated with

$$\sigma_1^n := \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \qquad n \geqslant 0,$$

we find that

$$\begin{aligned}
\mu \,\|\nabla(\tilde{\varphi}_k - \tilde{\varphi}_m)\|^2_{L_2(\Omega)} &\leqslant \int_{\Omega_k} \left[\sigma_1^k \varphi_k - \sum_{i=1}^{N} \tilde{\alpha}_i \frac{\partial \varphi_k}{\partial x_i} - \alpha_0 \varphi_k \right] \varphi_k \\
&\quad + \int_{\Omega_m} \left[\sigma_1^m \varphi_m - \sum_{i=1}^{N} \tilde{\alpha}_i \frac{\partial \varphi_m}{\partial x_i} - \alpha_0 \varphi_m \right] \varphi_m \\
&\quad -2 \int_{\Omega_m} \left[\sigma_1^k \varphi_k - \sum_{i=1}^{N} \tilde{\alpha}_i \frac{\partial \varphi_k}{\partial x_i} - \alpha_0 \varphi_k \right] \varphi_m \\
&\quad + \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \left\{ \frac{\partial \varphi_k}{\partial x_i} \varphi_k n_j + \frac{\partial \varphi_m}{\partial x_i} \varphi_m n_j - 2 \frac{\partial \varphi_k}{\partial x_i} \varphi_m n_j \right\},
\end{aligned} \tag{7.13}$$

where the function coefficients $\tilde{\alpha}_i \in \mathscr{C}(\bar{\Omega})$, $1 \leqslant i \leqslant N$, are those given by (2.6). Rearranging terms in (7.13) gives

$$\begin{aligned}
\mu\,\|\nabla(\tilde{\varphi}_k - \tilde{\varphi}_m)\|^2_{L_2(\Omega)} \leqslant{}& \sigma_1^k \int_{\Omega_k} \varphi_k(\varphi_k - \tilde{\varphi}_m) + (\sigma_1^m - \sigma_1^k) \int_{\Omega_m} \varphi_m^2 \\
&+ \sigma_1^k \int_{\Omega_m} \varphi_m(\varphi_m - \varphi_k) + \sum_{i=1}^N \int_{\Omega_k} \tilde{\alpha}_i (\tilde{\varphi}_m - \varphi_k) \frac{\partial \varphi_k}{\partial x_i} \\
&+ \sum_{i=1}^N \int_{\Omega_m} \tilde{\alpha}_i \varphi_m \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \\
&+ \int_{\Omega_k} \alpha_0 \varphi_k (\tilde{\varphi}_m - \varphi_k) + \int_{\Omega_m} \alpha_0 \varphi_m (\varphi_k - \varphi_m) \\
&+ \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \left\{ (\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} + \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k) \right\} n_j.
\end{aligned} \tag{7.14}$$

Now, we shall estimate each of the terms in the right hand side of (7.14). Note that, thanks to (7.4), the following estimates hold

$$\|\tilde{\varphi}_n\|_{L_2(\Omega)} \leqslant 1, \qquad \|\nabla \tilde{\varphi}_n\|_{L_2(\Omega)} \leqslant 1, \qquad n \geqslant 0. \tag{7.15}$$

Moreover, σ_1^n, $n \geqslant 1$, is an increasing sequence bounded above by σ_1^0, by construction. Using this feature together with Hölder's inequality and (7.15) yields

$$\left| \sigma_1^k \int_{\Omega_k} \varphi_k(\varphi_k - \tilde{\varphi}_m) \right| \leqslant |\sigma_1^0|\; \|\tilde{\varphi}_k - \tilde{\varphi}_m\|_{L_2(\Omega)}, \tag{7.16}$$

$$\left| (\sigma_1^m - \sigma_1^k) \int_{\Omega_m} \varphi_m^2 \right| \leqslant |\sigma_1^m - \sigma_1^k|, \tag{7.17}$$

$$\left| \sigma_1^k \int_{\Omega_m} \varphi_m(\varphi_m - \varphi_k) \right| \leqslant |\sigma_1^0|\; \|\tilde{\varphi}_m - \tilde{\varphi}_k\|_{L_2(\Omega)}, \tag{7.18}$$

$$\left| \sum_{i=1}^N \int_{\Omega_k} \tilde{\alpha}_i (\tilde{\varphi}_m - \varphi_k) \frac{\partial \varphi_k}{\partial x_i} \right| \leqslant \left(\sum_{i=1}^N \|\tilde{\alpha}_i\|_{L_\infty(\Omega)} \right) \|\tilde{\varphi}_m - \tilde{\varphi}_k\|_{L_2(\Omega)}, \tag{7.19}$$

$$\left| \int_{\Omega_k} \alpha_0 \varphi_k (\tilde{\varphi}_m - \varphi_k) \right| \leqslant \|\alpha_0\|_{L_\infty(\Omega)}\; \|\tilde{\varphi}_m - \tilde{\varphi}_k\|_{L_2(\Omega)}, \tag{7.20}$$

and

$$\left| \int_{\Omega_m} \alpha_0 \varphi_m (\varphi_k - \varphi_m) \right| \leqslant \|\alpha_0\|_{L_\infty(\Omega)} \, \|\tilde{\varphi}_m - \tilde{\varphi}_k\|_{L_2(\Omega)} \,. \tag{7.21}$$

In order to estimate the integrals over Γ_1 one should remember that for any $n \geqslant 1$

$$\partial_\nu \varphi_n + b\varphi_n = 0 \qquad \text{on} \quad \Gamma_1 \,,$$

where

$$\nu_i := \sum_{j=1}^{N} \alpha_{ij} \, n_j \,, \qquad 1 \leqslant i \leqslant N.$$

Thus, for any natural number $n \geqslant 0$ we have

$$\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} n_j = \sum_{i=1}^{N} \nu_i \frac{\partial \varphi_n}{\partial x_i} = \langle \nabla \varphi_n \,, \nu \rangle = \partial_\nu \varphi_n = -b\varphi_n \,,$$

and hence

$$\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k) \, n_j = -b(\varphi_m - \varphi_k).$$

Therefore,

$$\left| \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} (\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} n_j \right| = \left| \int_{\Gamma_1} b\varphi_k (\varphi_m - \varphi_k) \right|$$
$$\leqslant \|b\|_{L_\infty(\Gamma_1)} \, \|\varphi_k|_{\Gamma_1}\|_{L_2(\Gamma_1)} \, \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \,, \tag{7.22}$$

and

$$\left| \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k) \, n_j \right| = \left| \int_{\Gamma_1} b \, \varphi_m (\varphi_k - \varphi_m) \right|$$
$$\leqslant \|b\|_{L_\infty(\Gamma_1)} \, \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)} \, \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \,. \tag{7.23}$$

It remains to estimate the term

$$I_{mk} := \sum_{i=1}^{N} \int_{\Omega_m} \tilde{\alpha}_i \varphi_m \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m). \tag{7.24}$$

Since $\tilde{\alpha}_i \in \mathscr{C}(\bar{\Omega})$ in order to perform an integration by parts in (7.24) we must approach each of the coefficients $\tilde{\alpha}_i$, $1 \leqslant i \leqslant N$, by a sequence of smooth coefficients, say α_i^n, $n \geqslant 1$.

Fix $\delta > 0$ and consider the δ-neighborhood of Ω

$$\Omega_\delta := \bar{\Omega} + B_\delta(0).$$

For each $1 \leqslant i \leqslant N$, let $\hat{\alpha}_i$ be a continuous extension of $\tilde{\alpha}_i$ to $\mathbf{R}^N$ such that

$$\hat{\alpha}_i \in \mathscr{C}_c(\Omega_\delta), \qquad \|\hat{\alpha}_i\|_{L_\infty(\mathbf{R}^N)} = \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}. \tag{7.25}$$

Now, consider the function

$$\rho(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geqslant 1, \end{cases}$$

and the associated *approximation of the identity*

$$\rho_n := \left(\int_{\mathbf{R}^N} \rho \right)^{-1} n^N \rho(n\,\cdot), \qquad n \in \mathbf{N}.$$

Note that for each $n \geqslant 1$ the function ρ_n satisfies

$$\rho_n \in \mathscr{C}_c^\infty(\mathbf{R}^N), \qquad \operatorname{supp} \rho_n \subset B_{\frac{1}{n}}(0), \qquad \rho_n \geqslant 0, \qquad \|\rho_n\|_{L_1(\mathbf{R}^N)} = 1.$$

Then, for each $1 \leqslant i \leqslant N$ the new sequence

$$\alpha_i^n := \rho_n * \hat{\alpha}_i\,, \qquad n \geqslant 1,$$

is of class $\mathscr{C}_c^\infty(\mathbf{R}^N)$ and it converges to $\hat{\alpha}_i$ uniformly on any compact subset of $\mathbf{R}^N$ (e.g. Theorem 8.1.3 of [18]). In particular,

$$\lim_{n\to\infty} \|\alpha_i^n|_\Omega - \tilde{\alpha}_i\|_{L_\infty(\Omega)} = 0, \qquad 1 \leqslant i \leqslant N, \tag{7.26}$$

since $\hat{\alpha}_i|_\Omega = \tilde{\alpha}_i$. Moreover, thanks to (7.25), it follows from Young's inequality that

$$\|\alpha_i^n\|_{L_\infty(\mathbf{R}^N)} \leqslant \|\rho_n\|_{L_1(\mathbf{R}^N)} \|\hat{\alpha}_i\|_{L_\infty(\mathbf{R}^N)} = \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}, \qquad 1 \leqslant i \leqslant N, \quad n \geqslant 1, \tag{7.27}$$

and

$$\left\| \frac{\partial \alpha_i^n}{\partial x_i} \right\|_{L_\infty(\mathbf{R}^N)} \leqslant \left\| \frac{\partial \rho_n}{\partial x_i} \right\|_{L_1(\mathbf{R}^N)} \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}, \qquad 1 \leqslant i \leqslant N, \quad n \geqslant 1, \tag{7.28}$$

since

$$\frac{\partial \alpha_i^n}{\partial x_i} = \frac{\partial \rho_n}{\partial x_i} * \hat{\alpha}_i , \qquad 1 \leqslant i \leqslant N, \quad n \geqslant 1.$$

On the other hand, for each $1 \leqslant i \leqslant N$ and $n \geqslant 1$

$$\left\| \frac{\partial \rho_n}{\partial x_i} \right\|_{L_1(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} \rho \right)^{-1} n \left\| \frac{\partial \rho}{\partial x_i} \right\|_{L_1(\mathbf{R}^N)},$$

and hence (7.28) implies

$$\left\| \frac{\partial \alpha_i^n}{\partial x_i} \right\|_{L_\infty(\mathbf{R}^N)} \leqslant \left(\int_{\mathbf{R}^N} \rho \right)^{-1} n \left\| \frac{\partial \rho}{\partial x_i} \right\|_{L_1(\mathbf{R}^N)} \|\tilde{\alpha}_i\|_{L_\infty(\Omega)} , \qquad 1 \leqslant i \leqslant N, \quad n \geqslant 1. \tag{7.29}$$

Now, going back to (7.24) we find that for each $n \geqslant 1$

$$I_{mk} := \sum_{i=1}^N \int_{\Omega_m} (\tilde{\alpha}_i - \alpha_i^n) \, \varphi_m \, \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) + \sum_{i=1}^N \int_{\Omega_m} \alpha_i^n \varphi_m \, \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m). \tag{7.30}$$

We now estimate each of the terms in the right hand side of (7.30). Applying Hölder inequality it is easily seen that

$$\begin{aligned} &\left| \sum_{i=1}^N \int_{\Omega_m} (\tilde{\alpha}_i - \alpha_i^n) \, \varphi_m \, \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \right| \\ &\qquad \leqslant \left(\sum_{i=1}^N \|\tilde{\alpha}_i - \alpha_i^n\|_{L_\infty(\Omega)} \right) \|\tilde{\varphi}_m\|_{L_2(\Omega)} \, \|\nabla(\tilde{\varphi}_k - \tilde{\varphi}_m)\|_{L_2(\Omega)} \\ &\qquad \leqslant 2 \sum_{i=1}^N \|\tilde{\alpha}_i - \alpha_i^n\|_{L_\infty(\Omega)} . \end{aligned}$$

Moreover, integrating by parts gives

$$\begin{aligned} &\sum_{i=1}^N \int_{\Omega_m} \alpha_i^n \varphi_m \, \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \\ &\qquad = -\sum_{i=1}^N \int_{\Omega_m} (\varphi_k - \varphi_m) \, \frac{\partial}{\partial x_i} (\alpha_i^n \varphi_m) + \sum_{i=1}^N \int_{\Gamma_1} \alpha_i^n \varphi_m (\varphi_k - \varphi_m) \, n_i , \end{aligned}$$

and hence,

$$\left| \sum_{i=1}^{N} \int_{\Omega_m} \alpha_i^n \varphi_m \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \right|$$
$$\leqslant \left(\sum_{i=1}^{N} \|\alpha_i^n\|_{L_\infty(\mathbf{R}^N)} \right) \|\nabla \tilde{\varphi}_m\|_{L_2(\Omega)} \|\tilde{\varphi}_k - \tilde{\varphi}_m\|_{L_2(\Omega)}$$
$$+ \left(\sum_{i=1}^{N} \left\| \frac{\partial \alpha_i^n}{\partial x_i} \right\|_{L_\infty(\mathbf{R}^N)} \right) \|\tilde{\varphi}_m\|_{L_2(\Omega)} \|\tilde{\varphi}_k - \tilde{\varphi}_m\|_{L_2(\Omega)}$$
$$+ \left(\sum_{i=1}^{N} \|\alpha_i^n\|_{L_\infty(\mathbf{R}^N)} \right) \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)} \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}.$$

Thus, substituting these estimates into (7.30) and using (7.15), (7.27), and (7.29) we find that for any $n \geqslant 1$

$$|I_{mk}| \leqslant 2 \sum_{i=1}^{N} \|\tilde{\alpha}_i - \alpha_i^n\|_{L_\infty(\Omega)} + \sum_{i=1}^{N} \|\tilde{\alpha}_i\|_{L_\infty(\Omega)} \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)} \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)}$$
$$+ \sum_{i=1}^{N} \left(1 + \left(\int_{\mathbf{R}^N} \rho \right)^{-1} n \left\| \frac{\partial \rho}{\partial x_i} \right\|_{L_1(\mathbf{R}^N)} \right) \|\tilde{\alpha}_i\|_{L_\infty(\Omega)} \|\tilde{\varphi}_k - \tilde{\varphi}_m\|_{L_2(\Omega)}.$$

Now, fix $\varepsilon > 0$. Thanks to (7.26) there exists $n \geqslant 1$ such that

$$2 \sum_{i=1}^{N} \|\tilde{\alpha}_i - \alpha_i^n\|_{L_\infty(\Omega)} \leqslant \frac{\varepsilon}{4}.$$

Hence, thanks to (7.5) and (7.11), there exists $n_0 \geqslant 1$ such that for any $n_0 \leqslant k \leqslant m$

$$|I_{mk}| \leqslant \frac{\varepsilon}{2}. \tag{7.31}$$

Therefore, substituting (7.16)–(7.21) and (7.22)–(7.23) into (7.14) and using (7.2), (7.5), (7.11), and (7.31), it is easily seen that there exists $k_0 \geqslant n_0$ such that for any $k_0 \leqslant k \leqslant m$

$$\mu \|\nabla(\tilde{\varphi}_k - \tilde{\varphi}_m)\|_{L_2(\Omega)}^2 \leqslant \varepsilon.$$

This shows that $\tilde{\varphi} \in H^1(\Omega)$ and completes the proof of (7.12). Note that

$$\|\tilde{\varphi}\|_{H^1(\Omega)} = \lim_{n \to \infty} \|\tilde{\varphi}_n\|_{H^1(\Omega)} = \lim_{n \to \infty} \|\varphi_n\|_{H^1(\Omega_n)} = 1. \tag{7.32}$$

Moreover, if γ^1 stands for the trace operator of $H^1(\Omega)$ on Γ_1, then

$$\|\varphi_n|_{\Gamma_1} - \tilde{\varphi}|_{\Gamma_1}\|_{L_2(\Gamma_1)} = \|\gamma^1(\tilde{\varphi}_n - \tilde{\varphi})\|_{L_2(\Gamma_1)} \leqslant \|\gamma^1\|_{\mathscr{L}(H^1(\Omega),\, L_2(\Gamma_1))} \|\tilde{\varphi}_n - \tilde{\varphi}\|_{H^1(\Omega)}$$

and hence, thanks to (7.12),

$$\lim_{n\to\infty} \|\varphi_n|_{\Gamma_1} - \tilde{\varphi}|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0.$$

Thus, thanks to (7.11) we find that

$$\tilde{\varphi}|_{\Gamma_1} = \varphi^*. \tag{7.33}$$

Set

$$\varphi := \tilde{\varphi}|_{\Omega_0}. \tag{7.34}$$

Since $\varphi_n|_{\Omega_0} = \tilde{\varphi}_n|_{\Omega_0}$, by (7.12) we have that $\varphi \in H^1(\Omega_0)$ and that

$$\lim_{n\to\infty} \|\varphi_n|_{\Omega_0} - \varphi\|_{H^1(\Omega_0)} = 0.$$

Moreover, thanks to (7.7) and (7.32),

$$\|\varphi\|_{H^1(\Omega_0)} = \|\tilde{\varphi}\|_{H^1(\Omega)} = 1. \tag{7.35}$$

Thus, thanks to (7.8), $\varphi > 0$ in Ω_0. We now show that φ is a weak solution of

$$\begin{cases} \mathscr{L}\varphi = \sigma_1^E \varphi & \text{in} \quad \Omega_0, \\ \mathscr{B}_0(b)\,\varphi = 0 & \text{on} \quad \partial\Omega_0, \end{cases} \tag{7.36}$$

where σ_1^E is the limit (7.2) and $\mathscr{B}_0(b)$ is the boundary operator defined by

$$\mathscr{B}_0(b)\,u := \begin{cases} u & \text{on} \quad \Gamma_0^0, \\ \partial_\nu u + bu & \text{on} \quad \Gamma_1. \end{cases}$$

Indeed, since $\tilde{\varphi} \in H^1(\Omega)$ and $\operatorname{supp} \tilde{\varphi} \subset \bar{\Omega}_0$ it follows from Theorem 6.4 that $\tilde{\varphi} \in H^1_{\Gamma_0^0}(\Omega_0)$, and hence $\varphi = \tilde{\varphi}|_{\Omega_0} \in H^1_{\Gamma_0^0}(\Omega_0)$. Now, pick

$$\xi \in \mathscr{C}_c^\infty(\Omega_0 \cup \Gamma_1).$$

Then, multiplying the equation

$$\mathscr{L}\varphi_n = \sigma_1^n \varphi_n \qquad \text{in} \quad \Omega_n, \qquad n \geqslant 1,$$

by ξ, integrating in Ω_n, applying the formula of integration by parts and taking into account that $\operatorname{supp}\xi \subset \Omega_0 \cup \Gamma_1$ gives

$$\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial \varphi_n}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 \varphi_n \xi$$
$$= \sigma_1^n \int_{\Omega_0} \varphi_n \xi + \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \xi \, n_j,$$

for each $n \geqslant 1$. Moreover, using

$$\partial_\nu \varphi_n + b\varphi_n = 0 \qquad \text{on} \quad \Gamma_1, \qquad n \geqslant 1,$$

yields

$$\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \xi n_j = \sum_{i=1}^{N} \nu_i \frac{\partial \varphi_n}{\partial x_i} \xi = \langle \nabla \varphi_n, \nu \rangle \, \xi = \partial_\nu \varphi_n \xi = -b\varphi_n \xi,$$

and hence for each $n \geqslant 1$ we find that

$$\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial \varphi_n}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 \varphi_n \xi = \sigma_1^n \int_{\Omega_0} \varphi_n \xi - \int_{\Gamma_1} b\varphi_n \xi. \tag{7.37}$$

Since $\tilde{\varphi}\,|_{\Gamma_1} = \varphi|_{\Gamma_1}$ and

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\Omega_0)} = 0, \qquad \lim_{n\to\infty} \|\varphi_n\,|_{\Gamma_1} - \tilde{\varphi}|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0,$$

passing to the limit as $n \to \infty$ in (7.37) the dominated convergence theorem implies

$$\sum_{i,j=1}^{N} \int_{\Omega_0} \alpha_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^{N} \int_{\Omega_0} \tilde{\alpha}_i \frac{\partial \varphi}{\partial x_i} \xi + \int_{\Omega_0} \alpha_0 \varphi \xi = \sigma_1^E \int_{\Omega_0} \varphi \xi - \int_{\Gamma_1} b\varphi \xi.$$

Thus, $\varphi \in H^1_{\Gamma_0^0}(\Omega_0)$ is a weak solution of (7.36) and for any $\omega > -\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)]$ the function $\varphi \in L_2^+(\Omega_0)$ provides us with a positive eigenfunction of $(\omega + \mathscr{L}_2)^{-1}$ associated with the eigenvalue $(\omega + \sigma_1^E)^{-1}$. Thanks to the proof of Theorem 12.1 of [3], the spectral radius of the operator $(\omega + \mathscr{L}_2)^{-1}$ in Ω_0 must equal $(\omega + \sigma_1^E)^{-1}$. On the other hand, thanks to the Krein–Rutman theorem (cf. [34, App. 3.2]),

$$\operatorname{spr}(\omega + \mathscr{L}_2)^{-1} = (\omega + \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)])^{-1}.$$

Therefore,

$$\sigma_1^E = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)].$$

Moreover, by the uniqueness of the principal eigenfunction $\varphi = \varphi_0$. This completes the proof of the theorem, since the argument is valid along any subsequence. ∎

Assume that Ω_0 is a proper stable subdomain of Ω with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ where $\Gamma_0^0 \cap \Gamma_1 = \emptyset$ and Γ_1 of class $\mathscr{C}^2$. Although the principal eigenvalue $\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)]$ might not be defined, since we are not requiring any regularity assumption on Γ_0^0, the proof of Theorem 7.1 still provides us with a sufficient condition for the existence of an eigenvalue associated with it there is a positive eigenfunction. Of course, except in the case when Γ_0^0 is of class $\mathscr{C}^2$, it is unknown whether or not the principal eigenvalue is unique. More precisely the following result is satisfied.

THEOREM 7.2. *Suppose* (7.1) *and* (2.7). *Let Ω_0 be a proper stable subdomain of Ω with boundary*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_0$ with Γ_1 of class $\mathscr{C}^2$, and assume that there exists a sequence Ω_n, $n \geqslant 1$, of bounded domains of $\mathbf{R}^N$ of class $\mathscr{C}^2$ converging to Ω_0 from the exterior and such that $\Omega_n \subset \Omega$, $n \geqslant 1$. Let $(\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \varphi_n)$ denote the principal eigen-pair associated with $(\mathscr{L}, \mathscr{B}_n(b), \Omega_n)$, where the principal eigenfunction is assumed to be normalized so that

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1, \qquad n \geqslant 1.$$

Then, there exists a subsequence of φ_n, $n \geqslant 1$, again labeled by n, and a function $\varphi \in H^1_{\Gamma_0^0}(\Omega_0)$ such that

$$\lim_{n\to\infty} \|\varphi_n|_{\Omega_0} - \varphi\|_{H^1(\Omega_0)} = 0,$$

and φ is a weak positive solution of (7.36).

Proof. Let Ω_{-1} be a proper subdomain of class $\mathscr{C}^2$ of Ω_0 such that

$$\partial\Omega_{-1} = \Gamma_0^{-1} \cup \Gamma_1,$$

where Γ_0^{-1} and Γ_1 are two disjoint open and closed subsets of $\partial\Omega_{-1}$. Then, since

$$\operatorname{dist}(\Gamma_1, \partial\Omega_{-1} \cap \Omega_n) = \operatorname{dist}(\Gamma_1, \Gamma_0^{-1}) > 0,$$

thanks to Proposition 3.2, for each $n \geqslant 1$ we have that

$$\sigma_1^n := \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] \leqslant \sigma_1^{\Omega_{-1}}[\mathscr{L}, \mathscr{B}(b, \Omega_{-1})],$$

and hence the limit

$$\sigma_1^E := \lim_{n \to \infty} \sigma_1^n$$

is well defined. Now, the construction of $\tilde{\varphi}$ such that

$$\lim_{m \to \infty} \|\tilde{\varphi}_{n_m} - \tilde{\varphi}\|_{H^1(\Omega)} = 0 \tag{7.38}$$

along some subsequence $\tilde{\varphi}_{n_m}$, $m \geqslant 1$, of $\tilde{\varphi}_n$, $n \geqslant 1$, can be carried out with the same argument of the proof of Theorem 7.1. But now to show that

$$\varphi := \tilde{\varphi}|_{\Omega_0} \in H^1_{\Gamma_0^0}(\Omega_0)$$

we cannot use Theorem 6.4 as in the proof of Theorem 7.1, since here we are not requiring Γ_0^0 to be smooth. Instead of that argument we use the following one. Since for each $m \geqslant 1$ we have $\varphi_{n_m} \in H^1(\Omega_{n_m})$ and Ω_{n_m}, $m \geqslant 1$, is a decreasing sequence of domains, necessarily

$$\tilde{\varphi}_{n_m} \in \bigcap_{k=1}^{m} H^1_{\Gamma_0^k}(\Omega_{n_k}), \qquad m \geqslant 1.$$

Hence, (7.38) implies

$$\tilde{\varphi} \in \bigcap_{k=1}^{\infty} H^1_{\Gamma_0^k}(\Omega_{n_k}) = H^1_{\Gamma_0^0}(\Omega_0),$$

since we are assuming that Ω_0 is stable. Therefore, we find from (7.38) that $\varphi \in H_{\Gamma_0^0}(\Omega_0)$. Finally, the same argument of the proof of Theorem 7.1 shows that φ is a weak positive solution of (7.36). This completes the proof. ▮

The following result provides us with the *continuous dependence from the interior*.

THEOREM 7.3. *Suppose* (7.1) *and* (2.7). *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathscr{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 *and* Γ_1 *are two disjoint open and closed subsets of* $\partial\Omega_0$ *and let* Ω_n, $n \geqslant 1$, *be a sequence of bounded domains of* $\mathbf{R}^N$ *of class* $\mathscr{C}^2$ *converging to* Ω_0

from the interior. For each $n \geqslant 0$*, let* $\mathscr{B}_n(b)$ *denote the boundary operator defined by*

$$\mathscr{B}_n(b)\, u := \begin{cases} u & \text{on} \quad \Gamma_0^n, \\ \partial_\nu u + bu & \text{on} \quad \Gamma_1, \end{cases}$$

where

$$\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1, \qquad n \geqslant 0,$$

and denote by $(\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \varphi_n)$ *the principal eigen-pair associated with* $(\mathscr{L}, \mathscr{B}_n(b), \Omega_n)$*, where the principal eigenfunction is assumed to be normalized so that*

$$\|\varphi_n\|_{H^1(\Omega_n)} = 1, \qquad n \geqslant 0.$$

Then, $\varphi_0 \in W^2_{\mathscr{B}_0(b)}(\Omega_0)$ *and*

$$\lim_{n\to\infty} \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)], \qquad \lim_{n\to\infty} \|\tilde{\varphi}_n - \varphi_0\|_{H^1(\Omega_0)} = 0,$$

where

$$\tilde{\varphi}_n := \begin{cases} \varphi_n & \text{in} \quad \Omega_n, \\ 0 & \text{in} \quad \Omega_0 \setminus \Omega_n, \end{cases} \qquad n \geqslant 0.$$

Proof. The existence and the uniqueness of the principal eigen-pairs

$$(\sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \varphi_n), \qquad n \geqslant 0,$$

is guaranteed by Theorem 12.1 of [3]. Set

$$\sigma_1^n := \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)], \qquad n \geqslant 0.$$

By construction, for each $n \geqslant 1$

$$\Omega_n \subset \Omega_{n+1} \subset \Omega_0,$$

and hence, thanks to Remark 6.2 and Proposition 3.2 we find that

$$\sigma_1^n \geqslant \sigma_1^{n+1} \geqslant \sigma_1^0, \qquad n \geqslant 1.$$

Therefore, the limit

$$\sigma_1^I := \lim_{n\to\infty} \sigma_1^n \tag{7.39}$$

is well defined. We have to show that $\sigma_1^I = \sigma_1^0$.

The proof of Theorem 7.1 can be easily adapted to show that there exists $\tilde{\varphi} \in H^1(\Omega_0)$, $\tilde{\varphi} > 0$, and a subsequence $\tilde{\varphi}_{n_m}$, $m \geqslant 1$, of $\tilde{\varphi}_n$, $n \geqslant 1$, such that

$$\lim_{m \to \infty} \|\tilde{\varphi}_{n_m} - \tilde{\varphi}\|_{H^1(\Omega_0)} = 0. \tag{7.40}$$

Since $\tilde{\varphi}_{n_m} \in H^1_{\Gamma_0^0}(\Omega_0)$ for each $m \geqslant 1$, (7.40) implies

$$\tilde{\varphi} \in H^1_{\Gamma_0^0}(\Omega_0).$$

Moreover, adapting the proof of Theorem 7.1 it is easily seen that $\tilde{\varphi}$ provides us with a weak positive solution of

$$\begin{cases} \mathscr{L}\tilde{\varphi} = \sigma_1^I \tilde{\varphi} & \text{in} \quad \Omega_0, \\ \mathscr{B}_0(b)\, \tilde{\varphi} = 0 & \text{on} \quad \partial\Omega_0. \end{cases} \tag{7.41}$$

Therefore,

$$\tilde{\varphi} = \varphi_0, \qquad \sigma_1^I = \sigma_1^0.$$

This completes the proof, since the same argument works out along any subsequence of φ_n, $n \geqslant 1$. ∎

As an immediate consequence, from Theorem 7.1 and Theorem 7.3 we find the continuous dependence of the principal eigenvalue with respect to the domain, which reads as follows.

THEOREM 7.4. *Suppose* (7.1) *and* (2.7). *Let* Ω_0 *be a proper subdomain of* Ω *with boundary of class* $\mathscr{C}^2$ *such that*

$$\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1,$$

where Γ_0^0 *and* Γ_1 *are two disjoint open and closed subsets of* $\partial\Omega_0$, *and let* Ω_n, $n \geqslant 1$, *be a sequence of bounded domains of* Ω *of class* $\mathscr{C}^2$ *converging to* Ω_0. *For each* $n \geqslant 0$, *let* $\mathscr{B}_n(b)$ *denote the boundary operator defined by*

$$\mathscr{B}_n(b)\, u := \begin{cases} u & \textit{on} \quad \Gamma_0^n, \\ \partial_\nu u + bu & \textit{on} \quad \Gamma_1, \end{cases}$$

where

$$\Gamma_0^n := \partial\Omega_n \setminus \Gamma_1, \qquad n \geqslant 0.$$

Then

$$\lim_{n \to \infty} \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}_n(b)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}_0(b)]. \tag{7.42}$$

Proof. By definition, there exist two sequences of bounded smooth domains, Ω_n^I and Ω_n^E, $n \geqslant 1$, such that Ω_n^I converges to Ω_0 from the interior, Ω_n^E does it from the exterior, as $n \to \infty$, and

$$\Omega_n^I \subset \Omega_0 \cap \Omega_n, \qquad \Omega_0 \cup \Omega_n \subset \Omega_n^E, \qquad n \geqslant 1.$$

In particular,

$$\Omega_n^I \subset \Omega_n \subset \Omega_n^E, \qquad n \geqslant 1,$$

and thanks to Proposition 3.2

$$\sigma_1^{\Omega_n^I}[\mathscr{L}, \mathscr{B}(b, \Omega_n^I)] \geqslant \sigma_1^{\Omega_n}[\mathscr{L}, \mathscr{B}(b, \Omega_n)] \geqslant \sigma_1^{\Omega_n^E}[\mathscr{L}, \mathscr{B}(b, \Omega_n^E)]. \tag{7.43}$$

On the other hand, due to Theorem 7.1 and Theorem 7.3 we find that

$$\lim_{n \to \infty} \sigma_1^{\Omega_n^E}[\mathscr{L}, \mathscr{B}(b, \Omega_n^E)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)],$$

$$\lim_{n \to \infty} \sigma_1^{\Omega_n^I}[\mathscr{L}, \mathscr{B}(b, \Omega_n^I)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)].$$

Therefore, (7.42) follows from (7.43). This completes the proof. ∎

Remark 7.5. If Ω_0 is a proper subdomain of Ω with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$, where $\Gamma_0^0 \cap \Gamma_1 = \emptyset$ and Γ_1 is of class $\mathscr{C}^2$ and we do not require any regularity on Γ_0^0 then for any sequence Ω_n, $n \geqslant 1$, of subdomains of Ω_0 of class $\mathscr{C}^2$ converging from the interior to Ω_0 as $n \to \infty$ the limit (7.39) is well defined and it provides us with an eigenvalue associated with it there is a weak positive solution of (7.41). The proof of this feature can be easily obtained adapting the proof of Theorem 7.3, and so we omit its details. It should be pointed out that, except in the case when Γ_0^0 is of class $\mathscr{C}^2$, σ_1^I might not be the unique eigenvalue with a weak positive eigenfunction. In fact, thanks to Theorem 7.2, if Ω_0 is assumed to be stable, then σ_1^E provides us with another eigenvalue associated with it there is a weak positive solution. Except when Γ_0^0 is of class $\mathscr{C}^2$, it is unknown whether or not

$$\sigma_1^I = \sigma_1^E. \tag{7.44}$$

It would be of great interest to characterize the class of all domains for which (7.44) holds, since these domains would provide us with the family of the domains possessing a unique principal eigenvalue. We conjecture that (7.44) is satisfied if, and only if, Ω_0 is stable in the sense of Definition 6.2.

It should be pointed out that Theorem 7.4 provides us with a substantial extension of Theorem 4.2 in [29] even in the very special case when

$\Gamma_1 = \emptyset$, since the regularity requirements on the coefficients of the differential operator here are substantially weaker than the regularity requirements of [29].

8. CONTINUOUS DEPENDENCE WITH RESPECT TO $b(x)$

In this section we analyze the continuous dependence of

$$\sigma_1(b) := \sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)] \tag{8.1}$$

with respect to the weight function $b(x)$ in the special case when ∂_ν is the conormal derivative with respect to $\mathscr{L}$, that is, when condition (2.7) is satisfied. To state our main result we need the following notation.

DEFINITION 8.1. By $\sigma(L_\infty(\Gamma_1), L_1(\Gamma_1))$ we denote the weak $*$ topology of $L_\infty(\Gamma_1)$. Thus, given a sequence $b_n \in \mathscr{C}(\Gamma_1)$, $n \geqslant 1$, it is said that

$$\lim_{n\to\infty} b_n = b \qquad \text{in} \quad \sigma(L_\infty(\Gamma_1), L_1(\Gamma_1)) \tag{8.2}$$

if

$$\lim_{n\to\infty} \int_{\Gamma_1} b_n \xi = \int_{\Gamma_1} b\xi$$

for each $\xi \in L_1(\Gamma_1)$.

THEOREM 8.2. *Suppose $\Gamma_1 \neq \emptyset$, (2.7) and (7.1), and let $b_n \in \mathscr{C}(\Gamma_1)$, $n \geqslant 1$, be an arbitrary sequence satisfying (8.2). For each $n \geqslant 1$ let φ_n denote the principal eigenfunction associated with $\sigma_1^n := \sigma_1(b_n)$ normalized so that*

$$\|\varphi_n\|_{H^1(\Omega)} = 1, \qquad n \geqslant 1. \tag{8.3}$$

Then

$$\lim_{n\to\infty} \sigma_1^n = \sigma_1(b), \qquad \lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\Omega)} = 0, \tag{8.4}$$

where φ stands for the principal eigenfunction associated with $\sigma_1(b)$, normalized so that $\|\varphi\|_{H^1(\Omega)} = 1$.

Proof. Thanks to the general assumptions on Ω, the existence and the uniqueness of the principal eigen-pairs (σ_1^n, φ_n), $n \geqslant 1$, and $(\sigma_1(b), \varphi)$ is guaranteed by Theorem 12.1 of [3]. Moreover, thanks to (8.2), we find from the Banach–Steinhaus theorem that b_n, $n \geqslant 1$, is uniformly bounded in $L_\infty(\Gamma_1)$ (e.g., Proposition III.12(iii) of [10]). In other words, there exists a constant $C > 0$ for which

$$\|b_n\|_{L_\infty(\Gamma_1)} \leqslant C, \qquad n \geqslant 1.$$

Thus, thanks to Proposition 3.5 we find that

$$\sigma_1(-C) \leqslant \sigma_1^n \leqslant \sigma_1(C), \qquad n \geqslant 1,$$

and therefore, there exists a subsequence of σ_1^n, $n \geqslant 1$, relabeled by n, such that

$$\sigma_1^\infty := \lim_{n \to \infty} \sigma_1^n \in \mathbf{R}$$

is well defined. Thanks to Theorem 12.1 of [3], for each $n \geqslant 1$

$$\varphi_n \in W^2_{\mathscr{B}(b_n)}(\Omega) \subset H^2(\Omega),$$

and hence, thanks to (8.3), there exists a subsequence of φ_n, relabeled by n, such that

$$\lim_{n \to \infty} \|\varphi_n - \varphi_\infty\|_{L_2(\Omega)} = 0 \tag{8.5}$$

for some $\varphi_\infty \in L_2(\Omega)$, since the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. As the previous argument is valid along any subsequence and the principal eigen-pair $(\sigma_1(b), \varphi)$ is unique, to complete the proof of the theorem it suffices to show that

$$\sigma_1^\infty = \sigma_1(b), \qquad \varphi_\infty = \varphi, \tag{8.6}$$

and that in fact

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{H^1(\Omega)} = 0. \tag{8.7}$$

Thanks to (8.5),

$$\lim_{n \to \infty} \varphi_n = \varphi_\infty \qquad \text{a.e. in} \quad \Omega,$$

and

$$\varphi_\infty \geqslant 0 \qquad \text{in} \quad \Omega,$$

since $\varphi_n > 0$ for each $n \geqslant 1$. As far as to the traces of the functions φ_n, $n \geqslant 1$, on Γ_1 concerns, the same argument of the proof of Theorem 7.1 shows that there exists a subsequence of φ_n, $n \geqslant 1$, again labeled by n, and a function $\varphi_* \in L_2(\Gamma_1)$ such that

$$\lim_{n\to\infty} \|\varphi_n|_{\Gamma_1} - \varphi_*\|_{L_2(\Gamma_1)} = 0. \tag{8.8}$$

In particular, there exists a constant $M > 0$ such that

$$\|\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leqslant M, \qquad n \geqslant 1. \tag{8.9}$$

We now adapt the argument of the proof of Theorem 7.1 to show that φ_n, $n \geqslant 1$, is a Cauchy sequence in $H^1(\Omega)$. Indeed, arguing as in the proof of Theorem 7.1 for any natural numbers $1 \leqslant k \leqslant m$ we have that for any $n \geqslant 1$

$$\begin{aligned}
\mu\, \|\nabla(\varphi_k - \varphi_m)\|^2_{L_2(\Omega)} \leqslant{}& \sigma_1^k \int_\Omega \varphi_k(\varphi_k - \varphi_m) + (\sigma_1^m - \sigma_1^k) \int_\Omega \varphi_m^2 \\
&+ \sigma_1^k \int_\Omega \varphi_m(\varphi_m - \varphi_k) + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i (\varphi_m - \varphi_k) \frac{\partial \varphi_k}{\partial x_i} \\
&+ \sum_{i=1}^N \int_\Omega (\tilde{\alpha}_i - \alpha_i^n)\, \varphi_m \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \\
&+ \sum_{i=1}^N \int_\Omega \alpha_i^n \varphi_m \frac{\partial}{\partial x_i} (\varphi_k - \varphi_m) \\
&+ \int_\Omega \alpha_0 \varphi_k (\varphi_m - \varphi_k) + \int_\Omega \alpha_0 \varphi_m (\varphi_k - \varphi_m) \\
&+ \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \left\{ (\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} + \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k) \right\} n_j,
\end{aligned} \tag{8.10}$$

where $\tilde{\alpha}_i \in \mathscr{C}(\bar{\Omega})$, $1 \leqslant i \leqslant N$, are the coefficients defined by (2.6) and for each $1 \leqslant i \leqslant N$, α_i^n, $n \geqslant 1$, is a sequence of class $\mathscr{C}_c^\infty(\mathbf{R}^N)$ such that

$$\lim_{n\to\infty} \|\alpha_i^n - \tilde{\alpha}_i\|_{L_\infty(\Omega)} = 0. \tag{8.11}$$

We suppose that α_i^n, $1 \leqslant i \leqslant N$, $n \geqslant 1$, has been constructed as in the proof of Theorem 7.1. Thanks to (8.3), for each $n \geqslant 1$ we have that

$$\|\varphi_n\|_{L_2(\Omega)} \leqslant 1, \qquad \|\nabla \varphi_n\|_{L_2(\Omega)} \leqslant 1, \tag{8.12}$$

and therefore, arguing as in the proof of Theorem 7.1 provides us with the following estimates

$$\left|\sigma_1^k \int_\Omega \varphi_k(\varphi_k-\varphi_m)\right| \leqslant \sup_{k\geqslant 1}\ \{|\sigma_1^k|\}\ \|\varphi_k-\varphi_m\|_{L_2(\Omega)}, \tag{8.13}$$

$$\left|(\sigma_1^m-\sigma_1^k)\int_\Omega \varphi_m^2\right| \leqslant |\sigma_1^m-\sigma_1^k|, \tag{8.14}$$

$$\left|\sigma_1^k \int_\Omega \varphi_m(\varphi_m-\varphi_k)\right| \leqslant \sup_{k\geqslant 1}\ \{|\sigma_1^k|\}\ \|\varphi_m-\varphi_k\|_{L_2(\Omega)}, \tag{8.15}$$

$$\left|\sum_{i=1}^N \int_\Omega \tilde{\alpha}_i(\varphi_m-\varphi_k)\frac{\partial\varphi_k}{\partial x_i}\right| \leqslant \left(\sum_{i=1}^N \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\right)\|\varphi_m-\varphi_k\|_{L_2(\Omega)}, \tag{8.16}$$

$$\left|\int_\Omega \alpha_0\varphi_k(\varphi_m-\varphi_k)\right| \leqslant \|\alpha_0\|_{L_\infty(\Omega)}\ \|\varphi_m-\varphi_k\|_{L_2(\Omega)}, \tag{8.17}$$

$$\left|\int_\Omega \alpha_0\varphi_m(\varphi_k-\varphi_m)\right| \leqslant \|\alpha_0\|_{L_\infty(\Omega)}\ \|\varphi_m-\varphi_k\|_{L_2(\Omega)}, \tag{8.18}$$

$$\left|\sum_{i=1}^N \int_\Omega (\tilde{\alpha}_i-\alpha_i^n)\,\varphi_m\frac{\partial}{\partial x_i}(\varphi_k-\varphi_m)\right| \leqslant 2\sum_{i=1}^N \|\tilde{\alpha}_i-\alpha_i^n\|_{L_\infty(\Omega)}, \tag{8.19}$$

and

$$\begin{aligned}\left|\sum_{i=1}^N \int_\Omega \alpha_i^n\varphi_m\frac{\partial}{\partial x_i}(\varphi_k-\varphi_m)\right| \\ \leqslant \sum_{i=1}^N \|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\ \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)}\ \|(\varphi_k-\varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \\ +\sum_{i=1}^N\left(1+\left(\int_{\mathbf{R}^N}\rho\right)^{-1} n\left\|\frac{\partial\rho}{\partial x_i}\right\|_{L_1(\mathbf{R}^N)}\right)\|\tilde{\alpha}_i\|_{L_\infty(\Omega)}\ \|\varphi_k-\varphi_m\|_{L_2(\Omega)}.\end{aligned} \tag{8.20}$$

On the other hand, for each $n\geqslant 1$ we have

$$\partial_\nu\varphi_n=-b_n\varphi_n \qquad \text{on} \quad \Gamma_1,$$

and hence, thanks to (2.7),

$$\sum_{i,j=1}^N \alpha_{ij}\frac{\partial\varphi_k}{\partial x_i}n_j=-b_k\varphi_k, \qquad \sum_{i,j=1}^N \alpha_{ij}\frac{\partial}{\partial x_i}(\varphi_m-\varphi_k)\,n_j=-b_m\varphi_m+b_k\varphi_k.$$

Thus,

$$\sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij}(\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} n_j = \int_{\Gamma_1} b_k \varphi_k (\varphi_m - \varphi_k), \tag{8.21}$$

$$\sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k)\, n_j = \int_{\Gamma_1} (b_k \varphi_k - b_m \varphi_m)\, \varphi_m . \tag{8.22}$$

Moreover, thanks to the fact that b_n, $n \geqslant 1$, is uniformly bounded and using (8.9) gives

$$\left| \int_{\Gamma_1} b_k \varphi_k (\varphi_m - \varphi_k) \right| \leqslant C \, \|(\varphi_m - \varphi_k)|_{\Gamma_1}\|_{L_2(\Gamma_1)}, \tag{8.23}$$

where $C > 0$ is a certain constant independent of k and m. Similarly,

$$\begin{aligned} \left| \int_{\Gamma_1} (b_k \varphi_k - b_m \varphi_m)\, \varphi_m \right| &\leqslant \left| \int_{\Gamma_1} b_k \varphi_m (\varphi_k - \varphi_m) \right| + \left| \int_{\Gamma_1} \varphi_m^2 (b_k - b_m) \right| \\ &\leqslant C \, \|(\varphi_m - \varphi_k)|_{\Gamma_1}\|_{L_2(\Gamma_1)} + \left| \int_{\Gamma_1} \varphi_m^2 (b_k - b_m) \right|. \end{aligned} \tag{8.24}$$

Moreover,

$$\begin{aligned} \left| \int_{\Gamma_1} \varphi_m^2 (b_k - b_m) \right| &\leqslant \left| \int_{\Gamma_1} (\varphi_m^2 - \varphi_*^2)(b_k - b_m) \right| + \left| \int_{\Gamma_1} \varphi_*^2 (b_k - b_m) \right| \\ &\leqslant 2 \sup_{k \geqslant 1} \{|b_k|\} \, \|\varphi_m + \varphi_*\|_{L_2(\Gamma_1)} \, \|\varphi_m - \varphi_*\|_{L_2(\Gamma_1)} \\ &\quad + \left| \int_{\Gamma_1} \varphi_*^2 (b_k - b_m) \right|, \end{aligned}$$

and hence, since b_k, $k \geqslant 1$, is uniformly bounded in $L_\infty(\Gamma_1)$ and it is a Cauchy sequence for the topology w^*, we find from (8.8) and (8.24) that for any $\varepsilon > 0$ there exists a natural number $n_0 \geqslant 1$ such that for all $k, m \geqslant n_0$

$$\left| \int_{\Gamma_1} (b_k \varphi_k - b_m \varphi_m)\, \varphi_m \right| \leqslant \frac{\varepsilon}{2}. \tag{8.25}$$

Finally, using (8.21)–(8.22) and substituting (8.13)–(8.20), (8.23), and (8.25) into (8.10) it is easily seen that there exists $k_0 \geqslant n_0$ such that

$$\mu \, \|\nabla(\varphi_k - \varphi_m)\|_{L_2(\Omega)}^2 \leqslant \varepsilon, \qquad k, m \geqslant k_0 .$$

Thanks to (8.5) this shows that

$$\lim_{n\to\infty} \|\varphi_n - \varphi_\infty\|_{H^1(\Omega)} = 0. \tag{8.26}$$

Moreover, by the continuity of the trace operator of $H^1(\Omega)$ on Γ_1, there exists a constant $C>0$ such that for each $n \geqslant 1$

$$\|(\varphi_n - \varphi_\infty)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leqslant C\, \|\varphi_n - \varphi_\infty\|_{H^1(\Omega)},$$

and hence (8.26) implies

$$\lim_{n\to\infty} \|\varphi_n|_{\Gamma_1} - \varphi_\infty|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0. \tag{8.27}$$

Therefore, thanks to (8.8),

$$\varphi_\infty|_{\Gamma_1} = \varphi_*.$$

Similarly, since $\varphi_n|_{\Gamma_0} = 0$ for each $n \geqslant 1$, by the continuity of the trace operator of $H^1(\Omega)$ on Γ_0 we find that

$$\varphi_\infty|_{\Gamma_0} = 0.$$

In particular,

$$\varphi_\infty \in H^1_{\Gamma_0}(\Omega).$$

Note that, thanks to (8.3), it follows from (8.26) that

$$\|\varphi_\infty\|_{H^1(\Omega)} = 1.$$

Thus, since $\varphi_n > 0$ for each $n \geqslant 1$,

$$\varphi_\infty > 0.$$

We now show that φ_∞ provides us with a weak solution of

$$\begin{cases} \mathscr{L}\varphi_\infty = \sigma_1^\infty \varphi_\infty & \text{in } \Omega, \\ \mathscr{B}(b)\,\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.28}$$

where $\sigma_1^\infty = \lim_{n\to\infty} \sigma_1^n$. We already know that $\varphi_\infty \in H^1_{\Gamma_0}(\Omega)$. Now, pick

$$\xi \in \mathscr{C}_c^\infty(\Omega \cup \Gamma_1).$$

Then, multiplying the equation

$$\mathscr{L}\varphi_n = \sigma_1^n \varphi_n \qquad \text{in} \quad \Omega, \qquad n \geqslant 1,$$

by ξ, integrating in Ω, applying the formula of integration by parts and taking into account that $\operatorname{supp} \xi \subset \Omega \cup \Gamma_1$ gives

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i \frac{\partial \varphi_n}{\partial x_i} \xi + \int_\Omega \alpha_0 \varphi_n \xi$$
$$= \sigma_1^n \int_\Omega \varphi_n \xi + \sum_{i,j=1}^N \int_{\Gamma_1} \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \xi n_j,$$

for each $n \geqslant 1$. Thus, it follows from

$$\partial_\nu \varphi_n = -b_n \varphi_n = 0 \qquad \text{on} \quad \Gamma_1, \qquad n \geqslant 1,$$

that for each $n \geqslant 1$

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial \varphi_n}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i \frac{\partial \varphi_n}{\partial x_i} \xi + \int_\Omega \alpha_0 \varphi_n \xi = \sigma_1^n \int_\Omega \varphi_n \xi - \int_{\Gamma_1} b_n \varphi_n \xi. \tag{8.29}$$

Moreover, it is easily seen from (8.26) and (8.27) that

$$\lim_{n \to \infty} \|(\varphi_n \xi - \varphi_\infty \xi)|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0,$$

and hence (8.2) implies that

$$\lim_{n \to \infty} \int_{\Gamma_1} b_n \varphi_n \xi = \int_{\Gamma_1} b \varphi_\infty \xi.$$

Thus, passing to the limit as $n \to \infty$ in (8.29) the dominated convergence theorem implies

$$\sum_{i,j=1}^N \int_\Omega \alpha_{ij} \frac{\partial \varphi_\infty}{\partial x_i} \frac{\partial \xi}{\partial x_j} + \sum_{i=1}^N \int_\Omega \tilde{\alpha}_i \frac{\partial \varphi_\infty}{\partial x_i} \xi + \int_\Omega \alpha_0 \varphi_\infty \xi = \sigma_1^\infty \int_\Omega \varphi_\infty \xi - \int_{\Gamma_1} b \varphi_\infty \xi,$$

and therefore $\varphi_\infty \in H^1_{\Gamma_0}(\Omega)$ is a weak positive solution of (8.28). This shows that for any $\omega > -\sigma_1(b)$ the function $\varphi_\infty \in L_2^+(\Omega)$ provides us with a positive eigenfunction of $(\omega + \mathscr{L}_2)^{-1}$ associated with the eigenvalue $(\omega + \sigma_1^\infty)^{-1}$. Thanks to the proof of Theorem 12.1 of [3], the spectral radius of the operator $(\omega+\mathscr{L}_2)^{-1}$ in Ω must equal $(\omega+\sigma_1^\infty)^{-1}$. On the other hand, thanks to the Krein–Rutman theorem (cf. [34, App. 3.2]),

$$\operatorname{spr}(\omega+\mathscr{L}_2)^{-1}=(\omega+\sigma_1(b))^{-1}.$$

Therefore,

$$\sigma_1^\infty=\sigma_1(b).$$

Moreover, by the uniqueness of the principal eigenfunction $\varphi_\infty=\varphi$. This completes the proof of the theorem, since the argument is valid along any subsequence. ∎

Remark 8.3. (a) Condition (8.2) holds provided

$$\lim_{n\to\infty}\|b_n-b\|_{L_\infty(\Omega)}=0.$$

(b) The proof of Theorem 8.2provides us with the existence of a principal eigenvalue for $(\mathscr{L},\mathscr{B}(b),\Omega)$, not necessarily unique, for a wide class of functions $b\in L_\infty(\Gamma_1)$, not necessarily continuous. Indeed, if b is the point-wise limit of a uniformly bounded sequence $b_n\in\mathscr{C}(\Gamma_1)$, $n\geqslant 1$, then it follows from the dominated Lebesgue convergence theorem that b_n is weakly $*$ convergent to b and the argument of the proof of Theorem 8.2 shows that σ_1^∞ is well defined and that associated with it there is a principal eigenfunction $\varphi_\infty\in H^1_{\Gamma_0}(\Omega)$.

9. ASYMPTOTIC BEHAVIOR OF $\sigma_1^\Omega[\mathscr{L},\mathscr{B}(b)]$ AS $\min_{\Gamma_1} b\nearrow\infty$

In this section we analyze the behavior of the principal eigenvalue (8.1) as $\min_{\Gamma_1} b\uparrow\infty$. The main result establishes that it converges to the principal eigenvalue of the Dirichlet problem in Ω. In obtaining this result we do not require (2.7) to be satisfied.

THEOREM 9.1. *Suppose $\Gamma_1\neq\emptyset$,*

$$\alpha_{ij}\in\mathscr{C}^1(\bar\Omega),\qquad \alpha_i\in\mathscr{C}(\bar\Omega),\qquad 1\leqslant i,j\leqslant N, \tag{9.1}$$

and let $b_n\in\mathscr{C}(\Gamma_1)$, $n\geqslant 1$, be an arbitrary sequence such that

$$\lim_{n\to\infty}\min_{\Gamma_1} b_n=\infty. \tag{9.2}$$

Set

$$\sigma_1^n := \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b_n)], \qquad n \geqslant 1,$$

and for each $n \geqslant 1$ let φ_n denote the principal eigenfunction associated with σ_1^n, normalized so that

$$\|\varphi_n\|_{H^1(\Omega)} = 1, \qquad n \geqslant 1. \tag{9.3}$$

Then

$$\lim_{n\to\infty} \sigma_1^n = \sigma_1^0, \qquad \lim_{n\to\infty} \|\varphi_n - \varphi_0\|_{H^1(\Omega)} = 0, \tag{9.4}$$

where (σ_1^0, φ_0) is the principal eigen-pair associated with the Dirichlet problem in Ω.

Proof. Since condition (9.2) is satisfied, without loss of generality we can assume that

$$\min_{\Gamma_1} b_n > 0, \qquad n \geqslant 1. \tag{9.5}$$

Now, combining Proposition 3.1 with Proposition 3.5 we find from (9.5) that

$$\sigma_1(0) < \sigma_1^n < \sigma_1^0, \qquad n \geqslant 1. \tag{9.6}$$

Thus, there exists $\sigma_1^\infty \in [\sigma_1(0), \sigma_1^0]$ and a subsequence of b_n, $n \geqslant 1$, relabeled by n, such that

$$\sigma_1^\infty := \lim_{n\to\infty} \sigma_1^n. \tag{9.7}$$

In the sequel we shall show that

$$\sigma_1^\infty = \sigma_1^0. \tag{9.8}$$

Thanks to Theorem 12.1 of [3] we have

$$\varphi_n \in W^2_{\mathscr{B}(b_n)}(\Omega) \subset H^2(\Omega), \qquad n \geqslant 1.$$

Thus, it follows from (9.3) and (9.6) that there exists a constant $C_1 > 0$ such that

$$\|(\sigma_1^n - \alpha_0)\, \varphi_n\|_{L_2(\Omega)} \leqslant |\sigma_1^n| \leqslant C_1, \qquad n \geqslant 1. \tag{9.9}$$

Thus, by the L^p estimates of Agmon *et al.* [2], there exists a constant $C_2 > 0$ such that

$$\|\varphi_n\|_{H^2(\Omega)} \leqslant C_2, \qquad n \geqslant 1. \tag{9.10}$$

Moreover, since $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, it follows from (9.3) that there exist a subsequence of φ_n, $n \geqslant 1$, relabeled by n, and a function $\varphi \in L_2(\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L_2(\Omega)} = 0. \tag{9.11}$$

Necessarily,

$$\lim_{n\to\infty} \varphi_n = \varphi \qquad \text{a.e. in} \quad \Omega,$$

and hence

$$\varphi \geqslant 0.$$

To complete the proof of the theorem it suffices to show that $(\sigma_1^\infty, \varphi) = (\sigma_1^0, \varphi_0)$ and that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\Omega)} = 0,$$

since the previous argument is valid along any subsequence of b_n, $n \geqslant 1$.

Let j_1, j_2 denote the compact injections

$$j_1 \colon W_2^{\frac{3}{2}}(\Gamma_1) \hookrightarrow L_2(\Gamma_1), \qquad j_2 \colon W_2^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L_2(\Gamma_1),$$

and

$$\gamma_1 \in \mathscr{L}(H^2(\Omega), W_2^{\frac{3}{2}}(\Gamma_1)), \qquad \gamma_2 \in \mathscr{L}(H^1(\Omega), W_2^{\frac{1}{2}}(\Gamma_1)),$$

the corresponding trace operators on Γ_1. Since $\varphi_n \in H^2(\Omega)$, for each $n \geqslant 1$ we have that

$$\varphi_n|_{\Gamma_1} \in W_2^{\frac{3}{2}}(\Gamma_1) \xrightarrow{j_1} L_2(\Gamma_1), \qquad \nabla\varphi_n|_{\Gamma_1} \in W_2^{\frac{1}{2}}(\Gamma_1) \xrightarrow{j_2} L_2(\Gamma_1),$$

and hence, it follows from (9.10) that

$$\begin{aligned}\|\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} &= \|j_1(\varphi_n|_{\Gamma_1})\|_{L_2(\Gamma_1)} \leqslant \|j_1\|\, \|\varphi_n|_{\Gamma_1}\|_{W_2^{3/2}(\Gamma_1)} \\ &\leqslant \|j_1\|\, \|\gamma_1\|\, \|\varphi_n\|_{H^2(\Omega)} \leqslant \|j_1\|\, \|\gamma_1\|\, C_2.\end{aligned}$$

Similarly,

$$\|\nabla\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leqslant \|j_2\|\,\|\nabla\varphi_n|_{\Gamma_1}\|_{W_2^{1/2}(\Gamma_1)} \leqslant \|j_2\|\,\|\gamma_2\|\,\|\nabla\varphi_n\|_{H^1(\Omega)}$$
$$\leqslant \|j_2\|\,\|\gamma_2\|\,\|\varphi_n\|_{H^2(\Omega)} \leqslant \|j_2\|\,\|\gamma_2\|\,C_2\,.$$

Therefore, there exists a constant $C_3>0$ such that

$$\|\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leqslant C_3\,, \qquad \|\nabla\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leqslant C_3\,, \qquad n\geqslant 1. \tag{9.12}$$

Now, setting

$$\beta_n := \min_{\Gamma_1}\, b_n\,, \qquad n\geqslant 1,$$

it follows from

$$\partial_\nu\varphi_n = -b_n\varphi_n \qquad \text{on} \quad \Gamma_1\,, \qquad n\geqslant 1,$$

that

$$(\partial_\nu\varphi_n)^2 = b_n^2\varphi_n^2 \geqslant \beta_n^2\varphi_n^2 \qquad \text{on} \quad \Gamma_1\,, \qquad n\geqslant 1,$$

and hence

$$\varphi_n^2|_{\Gamma_1} \leqslant \beta_n^{-2}(\partial_\nu\varphi_n)^2|_{\Gamma_1} \leqslant \beta_n^{-2}\,|\nu|^2\,|\nabla\varphi_n|_{\Gamma_1}|^2, \qquad n\geqslant 1,$$

where $|\cdot|$ stands for the euclidean norm of $\mathbf{R}^N$. Thus, applying (9.12) yields

$$\|\varphi_n|_{\Gamma_1}\|^2_{L_2(\Gamma_1)} \leqslant \beta_n^{-2}\,|\nu|^2\,\|\nabla\varphi_n|_{\Gamma_1}\|^2_{L_2(\Gamma_1)} \leqslant \beta_n^{-2}\,|\nu|^2\,C_3^2\,, \qquad n\geqslant 1,$$

and hence (9.2) implies

$$\lim_{n\to\infty}\,\|\varphi_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} = 0. \tag{9.13}$$

In particular,

$$\lim_{n\to\infty}\,\varphi_n|_{\Gamma_1} = 0 \qquad \text{a.e. in} \quad \Gamma_1\,.$$

On the other hand, by construction we have that $\varphi_n|_{\Gamma_0}=0$ for each $n\geqslant 1$. Therefore, we find from (9.13) that

$$\lim_{n\to\infty}\,\|\varphi_n|_{\partial\Omega}\|_{L_2(\partial\Omega)} = 0. \tag{9.14}$$

We now show that φ_n, $n \geqslant 1$, is a Cauchy sequence in $H^1(\Omega)$. Combining this fact with (9.11) gives

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{H^1(\Omega)} = 0, \qquad \|\varphi\|_{H^1(\Omega)} = 1. \tag{9.15}$$

Indeed, arguing as in the proof of Theorem 8.2 we find that for any $1 \leqslant k \leqslant m$ the estimate (8.10) is satisfied, as well as the estimates (8.12)–(8.20). Moreover,

$$\begin{aligned}\left| \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij}(\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} n_j \right| &\leqslant \sum_{i,j=1}^{N} \|\alpha_{ij}\|_{L_\infty(\Omega)} \|\nabla \varphi_k|_{\Gamma_1}\|_{L_2(\Gamma_1)} \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \\ &\leqslant C_3 \sum_{i,j=1}^{N} \|\alpha_{ij}\|_{L_\infty(\Omega)} \|(\varphi_k - \varphi_m)|_{\Gamma_1}\|_{L_2(\Gamma_1)},\end{aligned}$$

where we have used (9.12). Similarly,

$$\begin{aligned}&\left| \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k)\, n_j \right| \\ &\quad \leqslant \sum_{i,j=1}^{N} \|\alpha_{ij}\|_{L_\infty(\Omega)} \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)} \|\nabla(\varphi_m - \varphi_k)|_{\Gamma_1}\|_{L_2(\Gamma_1)} \\ &\quad \leqslant 2C_3 \sum_{i,j=1}^{N} \|\alpha_{ij}\|_{L_\infty(\Omega)} \|\varphi_m|_{\Gamma_1}\|_{L_2(\Gamma_1)}.\end{aligned}$$

Therefore, thanks to (9.14), for any $\varepsilon > 0$ there exists a natural number $n_0 \geqslant 0$ such that for any $k, m \geqslant n_0$ we have

$$\left| \sum_{i,j=1}^{N} \int_{\Gamma_1} \alpha_{ij} \left\{ (\varphi_k - \varphi_m) \frac{\partial \varphi_k}{\partial x_i} + \varphi_m \frac{\partial}{\partial x_i} (\varphi_m - \varphi_k) \right\} n_j \right| \leqslant \frac{\varepsilon}{2}. \tag{9.16}$$

Finally, substituting (8.13)–(8.20) and (9.16) into (8.10) it is easily seen that there exists $k_0 \geqslant n_0$ such that

$$\mu \|\nabla(\varphi_k - \varphi_m)\|^2_{L_2(\Omega)} \leqslant \varepsilon, \qquad k, m \geqslant k_0.$$

This completes the proof of (9.15), and in particular it shows that $\varphi \in H^1(\Omega)$ and that $\varphi > 0$.

We now ascertain the behavior of φ on $\partial\Omega$. Let j denote the compact injection

$$j\colon W_2^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L_2(\partial\Omega),$$

and

$$\gamma \in \mathscr{L}(H^1(\Omega), W_2^{\frac{1}{2}}(\partial\Omega)),$$

the trace operator on $\partial\Omega$. Since $\varphi_n - \varphi \in H^1(\Omega)$, for each $n \geqslant 1$ we have

$$(\varphi_n - \varphi)|_{\partial\Omega} \in W_2^{\frac{1}{2}}(\partial\Omega),$$

and hence

$$\begin{aligned} \|(\varphi_n - \varphi)|_{\partial\Omega}\|_{L_2(\partial\Omega)} &\leqslant \|j\| \, \|(\varphi_n - \varphi)|_{\partial\Omega}\|_{W_2^{1/2}(\partial\Omega)} = \|j\| \, \|\gamma(\varphi_n - \varphi)\|_{W_2^{1/2}(\partial\Omega)} \\ &\leqslant \|j\| \, \|\gamma\| \, \|\varphi_n - \varphi\|_{H^1(\Omega)}. \end{aligned}$$

Thus, (9.15) implies

$$\lim_{n \to \infty} \|(\varphi_n - \varphi)|_{\partial\Omega}\|_{L_2(\partial\Omega)} = 0,$$

and therefore, thanks to (9.14),

$$\gamma(\varphi) = \varphi|_{\partial\Omega} = 0.$$

In particular, (9.15) implies that

$$\varphi \in H_0^1(\Omega). \tag{9.17}$$

Now, the same argument used in the proofs of Theorem 7.1 and Theorem 8.2 shows that φ is a weak positive solution of

$$\begin{cases} \mathscr{L}\varphi = \sigma_1^\infty \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.18}$$

Therefore, by the uniqueness of the principal eigenpair, we find that

$$(\sigma_1^\infty, \varphi) = (\sigma_1^0, \varphi_0).$$

This completes the proof. ∎

As an immediate consequence of Theorem 9.1 we obtain the following result.

COROLLARY 9.2. *Suppose* (9.1). *Then,*

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{D}] = \sup_{b \in \mathscr{C}(\Gamma_1)} \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)]. \tag{9.19}$$

Proof. Thanks to Proposition 3.1,

$$\sup_{b\in\mathscr{C}(\Gamma_1)} \sigma_1^\Omega[\mathscr{L},\mathscr{B}(b)] \leqslant \sigma_1^\Omega[\mathscr{L},\mathscr{D}].$$

Moreover, due to Theorem 9.1 we have that

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathscr{L},\mathscr{B}(n)] = \sigma_1^\Omega[\mathscr{L},\mathscr{D}].$$

This completes the proof. ∎

10. VARYING THE MEASURE OF Ω

In this section we show that if the Lebesgue measure of Ω is sufficiently small and $\min_{\Gamma_1} b$ is sufficiently large, then $(\mathscr{L},\mathscr{B}(b),\Omega)$ satisfies the strong maximum principle, i.e., $\sigma_1^\Omega[\mathscr{L},\mathscr{B}(b)]>0$. This result is based upon the following generalization of Theorem 5.1 of [29].

THEOREM 10.1. *Suppose* (2.5) *and*

$$\Sigma_1^{\frac{1}{2}}\, |B_1|^{\frac{1}{N}}\, |\Omega|^{-\frac{1}{N}}\, \mu \geqslant |\tilde{\alpha}|_\infty, \tag{10.1}$$

where

$$B_1 := \{x\in\mathbf{R}^N : |x|<1\}, \qquad \Sigma_1 := \sigma_1^{B_1}[-\Delta,\mathscr{D}], \tag{10.2}$$

$|\cdot|$ *stands for the Lebesgue measure of* $\mathbf{R}^N$, $\mu>0$ *is the ellipticity constant of* $\mathscr{L}$ *in* Ω, *and*

$$\tilde{\alpha} := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N), \qquad |\tilde{\alpha}|_\infty := \sup_\Omega \left(\sum_{i=1}^N \tilde{\alpha}_i^2\right)^{\frac{1}{2}},$$

where $\tilde{\alpha}_i \in L_\infty(\Omega)$, $1\leqslant i\leqslant N$, *are the coefficients defined in* (2.6). *Then,*

$$\sigma_1^\Omega[\mathscr{L},\mathscr{D}] \geqslant \mu\Sigma_1\, |B_1|^{\frac{2}{N}}\, |\Omega|^{-\frac{2}{N}} - |\tilde{\alpha}|_\infty\, \Sigma_1^{\frac{1}{2}}\, |B_1|^{\frac{1}{N}}\, |\Omega|^{-\frac{1}{N}} + \inf_\Omega \alpha_0. \tag{10.3}$$

In particular,

$$\liminf_{|\Omega|\searrow 0} \sigma_1^\Omega[\mathscr{L},\mathscr{D}]\, |\Omega|^{\frac{2}{N}} \geqslant \mu\Sigma_1\, |B_1|^{\frac{2}{N}}. \tag{10.4}$$

Remark 10.2. (a) The estimate (10.1) is satisfied if $|\Omega|$ is sufficiently small. (b) Suppose $\mathscr{L}=-\Delta$. Then, thanks to the inequality of Faber [16] and Krahn [27], among all domains with a fixed Lebesgue measure, $|\Omega|$, the ball has the smallest principal eigenvalue under homogeneous Dirichlet boundary conditions. Thus, for any domain Ω the following estimate is satisfied

$$\sigma_1^{\Omega}[\mathscr{L},\mathscr{D}]\,|\Omega|^{\frac{2}{N}} \geqslant \Sigma_1\,|B_1|^{\frac{2}{N}}, \tag{10.5}$$

and therefore, the estimate (10.4) is optimal.

Proof of Theorem 10.1 Let $\varphi>0$ denote the principal eigenfunction associated with $\sigma_1^{\Omega}[\mathscr{L},\mathscr{D}]$. Then, multiplying the differential equation

$$\mathscr{L}\varphi=\sigma_1^{\Omega}[\mathscr{L},\mathscr{D}]\,\varphi$$

by φ, integrating in Ω, and applying the formula of integration by parts gives

$$\sigma_1^{\Omega}[\mathscr{L},\mathscr{D}]\int_\Omega \varphi^2=\sum_{i,j=1}^N\int_\Omega \alpha_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}+\sum_{i=1}^N\int_\Omega \tilde{\alpha}_i\varphi\frac{\partial\varphi}{\partial x_i}+\int_\Omega \alpha_0\varphi^2. \tag{10.6}$$

Due to the fact that $\mathscr{L}$ is strongly uniformly elliptic we find that

$$\sum_{i,j=1}^N\int_\Omega \alpha_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\geqslant \mu\int_\Omega |\nabla\varphi|^2. \tag{10.7}$$

Moreover, it follows from Hölder inequality that

$$\left|\sum_{i=1}^N\int_\Omega \tilde{\alpha}_i\varphi\frac{\partial\varphi}{\partial x_i}\right|=\left|\int_\Omega \varphi\langle\tilde{\alpha},\nabla\varphi\rangle\right|\leqslant\int_\Omega \varphi\,|\tilde{\alpha}|\,|\nabla\varphi|\leqslant|\tilde{\alpha}|_\infty\,\|\varphi\|_{L_2(\Omega)}\,\|\nabla\varphi\|_{L_2(\Omega)},$$

and hence

$$\sum_{i=1}^N\int_\Omega \tilde{\alpha}_i\varphi\frac{\partial\varphi}{\partial x_i}\geqslant-|\tilde{\alpha}|_\infty\,\|\varphi\|_{L_2(\Omega)}\,\|\nabla\varphi\|_{L_2(\Omega)}. \tag{10.8}$$

Thus, substituting (10.7) and (10.8) into (10.6) yields

$$\sigma_1^{\Omega}[\mathscr{L},\mathscr{D}]\geqslant\frac{\|\nabla\varphi\|_{L_2(\Omega)}}{\|\varphi\|_{L_2(\Omega)}}\left(\mu\frac{\|\nabla\varphi\|_{L_2(\Omega)}}{\|\varphi\|_{L_2(\Omega)}}-|\tilde{\alpha}|_\infty\right)+\inf_\Omega\,\alpha_0. \tag{10.9}$$

On the other hand, using the variational characterization of $\sigma_1^{\Omega}[-\Delta,\mathscr{D}]$, it follows from (10.5) that

$$\frac{\int_\Omega|\nabla\varphi|^2}{\int_\Omega\varphi^2}\geqslant\Sigma_1\,|B_1|^{\frac{2}{N}}\,|\Omega|^{-\frac{2}{N}}. \tag{10.10}$$

Thus, thanks to (10.10) and (10.1),

$$\mu \frac{\|\nabla \varphi\|_{L_2(\Omega)}}{\|\varphi\|_{L_2(\Omega)}} \geqslant \mu \Sigma_1^{\frac{1}{2}} |B_1|^{\frac{1}{N}} |\Omega|^{-\frac{1}{N}} \geqslant |\tilde{\alpha}|_{\infty}.$$

Finally, (10.3) follows by substituting (10.10) into (10.9). This completes the proof. ∎

Now, as an immediate consequence from Theorem 9.1 and Theorem 10.1, we obtain the following result.

COROLLARY 10.3. *Suppose* (9.1). *Then*

$$\liminf_{|\Omega| \searrow 0} \lim_{\substack{b \in \mathscr{C}(\Gamma_1) \\ \min_{\Gamma_1} b \nearrow \infty}} \sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)] \, |\Omega|^{\frac{2}{N}} \geqslant \mu \Sigma_1 \, |B_1|^{\frac{2}{N}}. \tag{10.11}$$

In particular, $\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)]$ can be as large as we wish by taking Ω with $|\Omega|$ sufficiently small and $b \in \mathscr{C}(\Gamma_1)$ with $\min_{\Gamma_1} b$ sufficiently large.

11. BLOWING UP THE AMPLITUDE OF A NONNEGATIVE POTENTIAL

In this section we introduce a class of nonnegative potentials $V \in L_{\infty}(\Omega)$ for which

$$\lim_{\lambda \nearrow \infty} \sigma_1^{\Omega}[\mathscr{L} + \lambda V, \mathscr{B}(b)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)], \tag{11.1}$$

where Ω_0 is the maximal open subset of Ω where V vanishes and $\mathscr{B}(b, \Omega_0)$ stands for the boundary operator introduced in (2.3). In the next section we shall use (11.1) to characterize the existence of principal eigenvalues for a large class of linear weighted boundary value problems of the form

$$\begin{cases} \mathscr{L}\varphi = \sigma W \varphi & \text{in} \quad \Omega, \\ \mathscr{B}(b)\, \varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{11.2}$$

where $W \in L_{\infty}(\Omega)$ is a sign indefinite potential. By a principal eigenvalue of (11.2) we mean a value of σ for which the problem admits a positive solution φ. As already pointed out in Section 1, the analysis of (11.2) is central from the point of view of the applications of the theory that we are developing to the applied sciences and engineering. Some substantially weaker versions of (11.1) were used by one of the authors in [15] to solve some classical open problems proposed by Simon in [35] within the context of the semi-classical analysis of Schrödinger operators. Therefore, (11.1) is of interest on its own right.

We now introduce the class of admissible potentials, denoted in the sequel by $\mathscr{A}$, for which (11.1) holds.

DEFINITION 11.1. It is said that $V \in L_\infty(\Omega)$, $V \geqslant 0$, is an admissible potential if there exist an open subset Ω_0 of Ω and a compact subset K of $\bar{\Omega}$ with Lebesgue measure zero such that

$$K \cap (\bar{\Omega}_0 \cup \Gamma_1) = \emptyset, \tag{11.3}$$

$$\Omega_+ := \{ x \in \Omega : V(x) > 0 \} = \Omega \setminus (\bar{\Omega}_0 \cup K), \tag{11.4}$$

and each of the following conditions is satisfied:

(a) Ω_0 possesses a finite number of components of class $\mathscr{C}^2$, say Ω_0^j, $1 \leqslant j \leqslant m$, such that $\bar{\Omega}_0^i \cap \bar{\Omega}_0^j = \emptyset$ if $i \neq j$, and

$$\operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0. \tag{11.5}$$

Thus, if we denote by Γ_1^i, $1 \leqslant i \leqslant n_1$, the components of Γ_1, then for each $1 \leqslant i \leqslant n_1$ either $\Gamma_1^i \subset \partial\Omega_0$ or $\Gamma_1^i \cap \partial\Omega_0 = \emptyset$. Moreover, if $\Gamma_1^i \subset \partial\Omega_0$, then Γ_1^i must be a component of $\partial\Omega_0$. Indeed, if $\Gamma_1^i \cap \partial\Omega_0 \neq \emptyset$ but Γ_1^i is not a component of $\partial\Omega_0$, then $\operatorname{dist}(\Gamma_1^i, \partial\Omega_0 \cap \Omega) = 0$.

(b) Let $\{i_1, ..., i_p\}$ denote the subset of $\{1, ..., n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_0 = \emptyset$ if and only if $j \in \{i_1, ..., i_p\}$. Then, V is bounded away from zero on any compact subset of

$$\Omega_+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}.$$

Note that if $\Gamma_1 \subset \partial\Omega_0$, then we are only imposing that V is bounded away from zero on any compact subset of Ω_+.

(c) Let Γ_0^i, $1 \leqslant i \leqslant n_0$, denote the components of Γ_0, and let $\{i_1, ..., i_q\}$ be the subset of $\{1, ..., n_0\}$ for which $(\partial\Omega_0 \cup K) \cap \Gamma_0^j \neq \emptyset$ if and only if $j \in \{i_1, ..., i_q\}$. Then, V is bounded away from zero on any compact subset of

$$\Omega_+ \cup \left[\bigcup_{j=1}^{q} \Gamma_0^{i_j} \setminus (\partial\Omega_0 \cup K) \right].$$

Note that if $(\partial\Omega_0 \cup K) \cap \Gamma_0 = \emptyset$, then we are only imposing that V is bounded away from zero on any compact subset of Ω_+.

(d) For any $\eta > 0$ there exist a natural number $\ell(\eta) \geqslant 1$ and $\ell(\eta)$ open subsets of $\mathbf{R}^N$, G_j^η, $1 \leqslant j \leqslant \ell(\eta)$, with $|G_j^\eta| < \eta$, $1 \leqslant j \leqslant \ell(\eta)$, such that

$$\bar{G}_i^\eta \cap \bar{G}_j^\eta = \varnothing \qquad \text{if} \quad i \neq j,$$

$$K \subset \bigcup_{j=1}^{\ell(\eta)} G_j^\eta,$$

and for each $1 \leqslant j \leqslant \ell(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathscr{C}^2$.

The family of all admissible potentials will be denoted by $\mathscr{A}$.

To state our main result we need the following concept.

Definition 11.2. Let Ω_0 be an open subset of Ω satisfying the requirements of Definition 11.1(a). Then, the principal eigenvalue of $(\mathscr{L}, \mathscr{B}(b, \Omega_0), \Omega_0)$ is defined through by

$$\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)] := \min_{1 \leqslant j \leqslant m} \sigma_1^{\Omega_0^j}[\mathscr{L}, \mathscr{B}(b, \Omega_0^j)].$$

Remark 11.3. Since Ω_0 is of class $\mathscr{C}^2$, it follows from (11.5) that each of the principal eigenvalues $\sigma_1^{\Omega_0^j}[\mathscr{L}, \mathscr{B}(b, \Omega_0^j)]$, $1 \leqslant j \leqslant m$, is well defined. This shows the consistency of Definition 11.2.

The main result of this section reads as follows.

Theorem 11.4. *Suppose* (7.1) *and* $V \in \mathscr{A}$. *Assume in addition that* (2.7) *holds on* $\Gamma_1 \cap \partial\Omega_0$. *Then,* (11.1) *is satisfied.*

Proof. Thanks to Proposition 3.2 and due to the fact that $V = 0$ in Ω_0, for each $1 \leqslant j \leqslant m$ and $\lambda \in \mathbf{R}$ we have that

$$\sigma_1^\Omega[\mathscr{L} + \lambda V, \mathscr{B}(b)] < \sigma_1^{\Omega_0^j}[\mathscr{L}, \mathscr{B}(b, \Omega_0^j)],$$

and hence

$$\sigma_1^\Omega[\mathscr{L} + \lambda V, \mathscr{B}(b)] < \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)]. \tag{11.6}$$

Moreover, thanks to Proposition 3.3, the map

$$\lambda \to P(\lambda) := \sigma_1^\Omega[\mathscr{L} + \lambda V, \mathscr{B}(b)]$$

is increasing. Thus, $\lim_{\lambda \nearrow \infty} P(\lambda)$ is well defined and due to (11.6)

$$\lim_{\lambda \nearrow \infty} P(\lambda) \leqslant \sigma_1^0 := \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)].$$

Therefore, to complete the proof of (11.1) it remains to show that for any $\varepsilon > 0$ there exists $\Lambda = \Lambda(\varepsilon) \in \mathbf{R}$ such that

$$P(\lambda) > \sigma_1^0 - \varepsilon \tag{11.7}$$

if $\lambda > \Lambda$. Note that (11.7) can be equivalently written in the form

$$\sigma_1^{\Omega}[\mathscr{L} + \lambda V - \sigma_1^0 + \varepsilon, \mathscr{B}(b)] > 0 \tag{11.8}$$

and that, thanks to Theorem 2.1, (11.8) is satisfied if, and only if,

$$(\mathscr{L} + \lambda V - \sigma_1^0 + \varepsilon, \mathscr{B}(b), \Omega) \tag{11.9}$$

possesses a positive strict supersolution for each $\lambda > \Lambda$. The remaining of the proof is devoted to the construction of a positive strict supersolution of (11.9) for λ large. In constructing it we need distinguishing between several different situations accordingly to the structure of the vanishing set of the potential, Ω_0. First we consider the simplest case when Ω_0 is connected and $K = \emptyset$. Then, we shall consider the general case.

Step 1. Suppose

$$m = 1, \qquad K = \emptyset.$$

Necessarily, either $\Gamma_0 \cap \partial\Omega_0 = \emptyset$ or $\Gamma_0 \cap \partial\Omega_0 \neq \emptyset$. Suppose

$$\Gamma_0 \cap \partial\Omega_0 = \emptyset. \tag{11.10}$$

For each $k \in \{0, 1\}$, let Γ_k^j, $1 \leqslant j \leqslant n_k$, denote the components of Γ_k. Let $\{i_1, ..., i_p\}$ denote the subset of $\{1, ..., n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_0 = \emptyset$ if and only if $j \in \{i_1, ..., i_p\}$. Note that, thanks to Definition 11.1(a), Γ_1^j is a component of $\partial\Omega_0$ for each $j \in \{1, ..., n_1\} \setminus \{i_1, ..., i_p\}$.

Fix $\varepsilon > 0$ and for each $\delta > 0$ sufficiently small consider the δ-neighborhoods

$$\Omega_\delta := (\Omega_0 + B_\delta) \cap \Omega, \tag{11.11}$$

$$\mathscr{N}_\delta^{0,j} := (\Gamma_0^j + B_\delta) \cap \Omega, \qquad 1 \leqslant j \leqslant n_0, \tag{11.12}$$

$$\mathscr{N}_\delta^{1,j} := (\Gamma_1^j + B_\delta) \cap \Omega, \qquad j \in \{i_1, ..., i_p\}, \tag{11.13}$$

where $B_\delta \subset \mathbf{R}^N$ is the ball of radius δ centered at the origin. Note that Ω_0 must possess at most finitely many holes, since it is of class $\mathscr{C}^2$. Thanks to (11.10), there exists $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$

$$\partial\Omega_\delta \setminus (\Gamma_1 \cap \partial\Omega_0) \subset \Omega_+, \qquad \bar{\Omega}_\delta \cap \bigcup_{j=1}^{n_0} \bar{\mathscr{N}}_\delta^{0,j} = \emptyset, \qquad \bigcup_{j=1}^{n_0} \bar{\mathscr{N}}_\delta^{0,j} \setminus \Gamma_0 \subset \Omega_+.$$

Moreover, since $\Gamma_k^j \cap \Gamma_\ell^i = \emptyset$ if $i \neq j$, there exists $\delta_1 \in (0, \delta_0)$ such that for each $0 < \delta < \delta_1$

$$\bar{\mathcal{N}}_\delta^{k,j} \cap \bar{\mathcal{N}}_\delta^{\ell,i} = \emptyset \qquad \text{if} \quad (i, \ell) \neq (j, k), \quad k, \ell \in \{0, 1\}.$$

Furthermore, since

$$\partial\Omega_0 \cap \bigcup_{j=1}^{p} \Gamma_1^{i_j} = \emptyset,$$

there exists $\delta_2 \in (0, \delta_1)$ such that for each $0 < \delta < \delta_2$

$$\bar{\Omega}_\delta \cap \bigcup_{j=1}^{p} \bar{\mathcal{N}}_\delta^{1, i_j} = \emptyset.$$

By construction, we have that Ω_0 is a proper subdomain of Ω_δ and that $\lim_{\delta \searrow 0} \Omega_\delta = \Omega_0$ in the sense of Definition 6.1. Thus, it follows from Proposition 3.2 and Theorem 7.1 that

$$\lim_{\delta \searrow 0} \sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)] = \sigma_1^{\Omega_0}[\mathcal{L}, \mathcal{B}(b, \Omega_0)] = \sigma_1^0,$$

$$\sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)] < \sigma_1^0, \qquad 0 < \delta < \delta_2,$$

since condition (2.7) is assumed to be satisfied on each of the components of $\Gamma_1 \cap \partial\Omega_0$. Therefore, there exists $\delta_3 \in (0, \delta_2)$ such that

$$\sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)] < \sigma_1^0 < \sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)] + \varepsilon \qquad \text{if} \quad 0 < \delta < \delta_3. \tag{11.14}$$

On the other hand, since $\lim_{\delta \searrow 0} |\mathcal{N}_\delta^{0,j}| = 0$ for each $1 \leqslant j \leqslant n_0$, it follows from Theorem 10.1 that

$$\lim_{\delta \searrow 0} \sigma_1^{\mathcal{N}_\delta^{0,j}}[\mathcal{L}, \mathcal{D}] = \infty, \qquad 1 \leqslant j \leqslant n_0,$$

and hence there exists $\delta_4 \in (0, \delta_3)$ such that for each $0 < \delta < \delta_4$

$$\sigma_1^{\mathcal{N}_\delta^{0,j}}[\mathcal{L}, \mathcal{D}] > \sigma_1^0, \qquad 1 \leqslant j \leqslant n_0. \tag{11.15}$$

Fix $\delta \in (0, \delta_4)$, and let φ_δ, ψ_δ^i, $i \in \{i_1, .., i_p\}$, and ξ_δ^j, $1 \leqslant j \leqslant n_0$, denote the principal eigenfunctions associated with $\sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)]$, $\sigma_1^{\mathcal{N}_\delta^{1,i}}[\mathcal{L}, \mathcal{B}(b, \mathcal{N}_\delta^{1,i})]$, $i \in \{i_1, ..., i_p\}$, and $\sigma_1^{\mathcal{N}_\delta^{0,j}}[\mathcal{L}, \mathcal{D}]$, $1 \leqslant j \leqslant n_0$, respectively, and consider the positive function $\Phi : \bar{\Omega} \to [0, \infty)$ defined by

$$\Phi := \begin{cases} \varphi_\delta & \text{in} \quad \bar{\Omega}_{\frac{\delta}{2}}, \\ \psi_\delta^{i_j} & \text{in} \quad \bar{\mathcal{N}}^{1, i_j}_{\frac{\delta}{2}}, \qquad 1 \leqslant j \leqslant p, \\ \xi_\delta^j & \text{in} \quad \bar{\mathcal{N}}^{0, j}_{\frac{\delta}{2}}, \qquad 1 \leqslant j \leqslant n_0, \\ \zeta_\delta & \text{in} \quad \bar{\Omega} \Big\backslash \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{p} \bar{\mathcal{N}}^{1, i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0, j}_{\frac{\delta}{2}} \right), \end{cases} \tag{11.16}$$

where ζ_δ is any regular positive extension of

$$\varphi_\delta \cup \bigcup_{j=1}^{p} \psi_\delta^{i_j} \cup \bigcup_{j=1}^{n_0} \xi_\delta^j$$

from

$$\bar{\Omega}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{p} \bar{\mathcal{N}}^{1, i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0, j}_{\frac{\delta}{2}}$$

to $\bar{\Omega}$ which is bounded away from zero in

$$\bar{\Omega} \Big\backslash \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{p} \bar{\mathcal{N}}^{1, i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0, j}_{\frac{\delta}{2}} \right).$$

Note that ζ_δ exists, since the functions $\varphi_\delta|_{\partial\Omega_{\delta/2} \cap \Omega}$, $\psi_\delta^{i_j}|_{\partial\mathcal{N}^{1, i_j}_{\delta/2} \backslash \Gamma_1^{i_j}}$, $1 \leqslant j \leqslant p$, and $\xi_\delta^j|_{\partial\mathcal{N}^{0, j}_{\delta/2} \backslash \Gamma_0^j}$, $1 \leqslant j \leqslant n_0$, are bounded away from zero. Moreover,

$$\Phi(x) > 0 \qquad \text{for each} \quad x \in \Omega.$$

If $\Gamma_1 \subset \partial\Omega_0$, then in the definition of $\Phi(x)$ we should delete the $\psi_\delta^{i_j}$'s.

To complete the proof of (11.1) in case (11.10) it remains to show that there exists $\Lambda = \Lambda(\varepsilon)$ such that Φ provides us with a strict supersolution of (11.9) for each $\lambda > \Lambda$. Indeed, since $V \geqslant 0$, it follows from (11.14) that in $\Omega_{\frac{\delta}{2}}$ the following estimate holds for any $\lambda > 0$

$$\begin{aligned} (\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \Phi &= (\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \varphi_\delta \\ &\geqslant (\sigma_1^{\Omega_\delta}[\mathcal{L}, \mathcal{B}(b, \Omega_\delta)] - \sigma_1^0 + \varepsilon)\, \varphi_\delta > 0. \end{aligned}$$

Similarly, thanks to (11.15), we find that for each $1 \leqslant j \leqslant n_0$ in $\mathcal{N}^{0, j}_{\frac{\delta}{2}}$ we have

$$(\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \Phi = (\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \xi_\delta^j \geqslant (\sigma_1^{\mathcal{N}^{0, j}_\delta}[\mathcal{L}, \mathcal{D}] - \sigma_1^0 + \varepsilon)\, \xi_\delta^j > 0.$$

Now, note that thanks to Definition 11.1(b) there exists a constant $\omega > 0$ such that

$$V \geqslant \omega > 0 \qquad \text{in} \quad \bar{\Omega} \Big\backslash \Big(\bar{\Omega}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0,j}_{\frac{\delta}{2}} \Big), \tag{11.17}$$

since

$$\bar{\Omega} \Big\backslash \Big(\Omega_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \mathcal{N}^{0,j}_{\frac{\delta}{2}} \Big) \subset \Omega_+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}.$$

Thus, thanks to (11.17), for each $1 \leqslant j \leqslant p$ in $\bar{\mathcal{N}}^{1,i_j}_{\frac{\delta}{2}}$ we have that

$$\begin{aligned}(\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \Phi &= (\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \psi_\delta^{i_j} \\ &\geqslant (\sigma_1^{\mathcal{N}^{1,i_j}_\delta}[\mathcal{L}, \mathcal{B}(b, \mathcal{N}^{1,i_j}_\delta)] - \sigma_1^0 + \varepsilon + \lambda\omega)\, \psi_\delta^{i_j} > 0,\end{aligned}$$

provided

$$\lambda > \max\{\omega^{-1}(\sigma_1^0 - \varepsilon - \sigma_1^{\mathcal{N}^{1,i_j}_\delta}[\mathcal{L}, \mathcal{B}(b, \mathcal{N}^{1,i_j}_\delta)]), 0\},$$

whereas in

$$\bar{\Omega} \Big\backslash \Big(\bar{\Omega}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{p} \bar{\mathcal{N}}^{1,i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0,j}_{\frac{\delta}{2}} \Big)$$

we have that

$$(\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \Phi = (\mathcal{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \zeta_\delta \geqslant (\mathcal{L} - \sigma_1^0 + \varepsilon)\, \zeta_\delta + \lambda\omega\zeta_\delta > 0$$

if $\lambda > 0$ is sufficiently large, since $(\mathcal{L} - \sigma_1^0 + \varepsilon)\, \zeta_\delta$ is independent of λ and ζ_δ is bounded away from zero. Finally, by construction,

$$\begin{aligned} \mathcal{B}(b)\, \Phi &= \mathcal{D}\xi_\delta^j = 0 && \text{on} \quad \Gamma_0^j, \qquad 1 \leqslant j \leqslant n_0, \\ \mathcal{B}(b)\, \Phi &= (\partial_\nu + b)\, \psi_\delta^{i_j} = 0 && \text{on} \quad \Gamma_1^{i_j}, \qquad 1 \leqslant j \leqslant p, \end{aligned}$$

and

$$\mathcal{B}(b)\, \Phi = (\partial_\nu + b)\, \varphi_\delta = 0 \qquad \text{on} \quad \partial\Omega_0 \cap \Gamma_1.$$

This completes the proof of the theorem when condition (11.10) is satisfied.

Now, suppose

$$\Gamma_0 \cap \partial\Omega_0 \neq \emptyset, \tag{11.18}$$

instead of (11.10). Let Γ_0^i, $1 \leqslant i \leqslant n_0$, denote the components of Γ_0, and let $\{i_1, ..., i_q\}$ be the subset of $\{1, ..., n_0\}$ for which $\partial\Omega_0 \cap \Gamma_0^j \neq \emptyset$ if and only if $j \in \{i_1, ..., i_q\}$.

Fix $\varepsilon > 0$ and given $\eta > 0$ sufficiently small consider the new support domain

$$G_\eta := \Omega \cup \left(\bigcup_{j=1}^{q} \Gamma_0^{i_j} + B_\eta \right).$$

Fix $\eta > 0$ and let $\tilde{\alpha}_{ij} = \tilde{\alpha}_{ji} \in \mathscr{C}^1(\bar{G}_\eta)$, $\tilde{\alpha}_i \in \mathscr{C}(\bar{G}_\eta)$, $\tilde{\alpha}_0 \in L_\infty(G_\eta)$ be regular extensions from $\bar{\Omega}$ to $\bar{G}_\eta$ of the coefficients $\alpha_{ij} = \alpha_{ji}$, α_i, α_0, $1 \leqslant i, j \leqslant N$, respectively. Now, consider the differential operator

$$\tilde{\mathscr{L}} := -\sum_{i,j=1}^{N} \tilde{\alpha}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \tilde{\alpha}_i \frac{\partial}{\partial x_i} + \tilde{\alpha}_0 \qquad \text{in} \quad G_\eta .$$

Since $\mathscr{L}$ is strongly uniformly elliptic in Ω with ellipticity constant $\mu > 0$, it is easily seen that there exists $\tilde{\eta} \in (0, \eta)$ such that $\tilde{\mathscr{L}}$ is strongly uniformly elliptic in $G_{\tilde{\eta}}$ with ellipticity constant $\frac{\mu}{2}$. Set

$$\tilde{\Omega} := G_{\tilde{\eta}} ,$$

and consider the new potential

$$\tilde{V} := \begin{cases} 1 & \text{in} \quad \tilde{\Omega} \setminus \Omega, \\ V & \text{in} \quad \Omega, \end{cases}$$

and the new boundary operator

$$\tilde{\mathscr{B}}(b) := \begin{cases} \mathscr{D} & \text{on} \quad \partial\tilde{\Omega} \setminus \Gamma_1 , \\ \partial_\nu + b, & \text{on} \quad \Gamma_1 . \end{cases}$$

From Definition 11.1(c) it is easily seen that $\tilde{V}$ belongs to the class $\tilde{\mathscr{A}}$ of admissible potentials in $\tilde{\Omega}$. Moreover, by construction

$$\tilde{\Omega}_0 = \Omega_0 , \qquad (\partial\tilde{\Omega} \setminus \Gamma_1) \cap \partial\tilde{\Omega}_0 = \emptyset.$$

Thus, condition (11.10) is satisfied for the new problem in $\tilde{\Omega}$, and hence there exist $\tilde{\Lambda} = \tilde{\Lambda}(\varepsilon)$ and a positive function $\tilde{\Phi}\colon \bar{\tilde{\Omega}} \to [0, \infty)$ with $\tilde{\Phi}(x) > 0$ for each $x \in \tilde{\Omega}$ which is an strict supersolution of

$$(\tilde{\mathscr{L}} + \lambda\tilde{V} - \tilde{\sigma}_1^0 + \varepsilon, \tilde{\mathscr{B}}(b), \tilde{\Omega})$$

for each $\lambda > \tilde{\Lambda}$, where

$$\tilde{\sigma}_1^0 := \sigma_1^{\tilde{\Omega}_0}[\tilde{\mathscr{L}}, \tilde{\mathscr{B}}(b, \tilde{\Omega}_0)] = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)] = \sigma_1^0.$$

Set

$$\Phi := \tilde{\Phi}|_{\bar{\Omega}}.$$

In Ω we have that for each $\lambda > \tilde{\Lambda}$

$$(\mathscr{L} + \lambda V - \sigma_1^0 + \varepsilon)\, \Phi = (\tilde{\mathscr{L}} + \lambda \tilde{V} - \tilde{\sigma}_1^0 + \varepsilon)\, \tilde{\Phi} \geqslant 0.$$

Moreover,

$$\Phi(x) = \tilde{\Phi}(x) > 0 \qquad \text{for each} \quad x \in \bigcup_{j=1}^{q} \Gamma_0^{i_j}, \tag{11.19}$$

since $\bigcup_{j=1}^{q} \Gamma_0^{i_j} \subset \tilde{\Omega}$, and

$$(\partial_\nu + b)\, \Phi|_{\Gamma_1} = (\partial_\nu + b)\, \tilde{\Phi}|_{\Gamma_1} \geqslant 0,$$

$$\Phi = \tilde{\Phi} = 0 \qquad \text{on} \quad \Gamma_0 \Big\backslash \bigcup_{j=1}^{q} \Gamma_0^{i_j}.$$

Thus, $\mathscr{B}(b)\, \Phi > 0$ on $\partial\Omega$, and therefore Φ provides us with a positive strict supersolution of (11.9) for each $\lambda > \tilde{\Lambda}$. This completes the proof of the theorem when Ω_0 is connected and $K = \emptyset$.

Step 2. Now, suppose that we are working under the general assumptions of the theorem and that in addition

$$\Gamma_0 \cap (\partial\Omega_0 \cup K) = \emptyset. \tag{11.20}$$

Then, (11.3) implies

$$K \subset \Omega, \qquad \bar{\Omega}_0 \subset \Omega \cup \Gamma_1, \qquad K \cap \bar{\Omega}_0 = \emptyset. \tag{11.21}$$

In particular,

$$\operatorname{dist}(\Gamma_0, \bar{\Omega}_0 \cup K) > 0, \qquad \operatorname{dist}(\Gamma_1, K) > 0, \qquad \operatorname{dist}(K, \bar{\Omega}_0) > 0. \tag{11.22}$$

Let Ω_0^i, $1 \leqslant i \leqslant m$, denote the components of Ω_0 and set

$$\sigma_1^i := \sigma_1^{\Omega_0^i}[\mathscr{L}, \mathscr{B}(b, \Omega_0^i)], \qquad 1 \leqslant i \leqslant m.$$

Without loss of generality we can assume that

$$\sigma_1^i \leqslant \sigma_1^{i+1}, \qquad 1 \leqslant i \leqslant m-1.$$

Fix $\eta > 0$. Thanks to Definition 11.1(d), there exist a natural number $\ell(\eta) \geqslant 1$ and $\ell(\eta)$ open sets $G_j^\eta \subset \mathbf{R}^N$, $1 \leqslant j \leqslant \ell(\eta)$, with $|G_j^\eta| < \eta$, $1 \leqslant j \leqslant \ell(\eta)$, such that

$$K \subset \bigcup_{j=1}^{\ell(\eta)} (G_j^\eta \cap \bar{\Omega}) \qquad \text{and} \qquad \bar{G}_i^\eta \cap \bar{G}_j^\eta = \varnothing \qquad \text{if} \quad i \neq j,$$

and for each $1 \leqslant j \leqslant \ell(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathscr{C}^2$. Thanks to (11.21), we can choose the G_j^η's so that

$$K \subset \bigcup_{j=1}^{\ell(\eta)} \bar{G}_j^\eta \subset \Omega, \qquad \bigcup_{j=1}^{\ell(\eta)} \bar{G}_j^\eta \cap \bar{\Omega}_0 = \varnothing. \tag{11.23}$$

Indeed, since

$$\operatorname{dist}(K, \bar{\Omega}_0 \cup \Gamma_0 \cup \Gamma_1) > 0,$$

there exists an open set G such that

$$K \subset G, \qquad \bar{G} \subset \Omega, \qquad \bar{G} \cap \bar{\Omega}_0 = \varnothing.$$

Hence, to get (11.23) it suffices to take $G \cap G_j^\eta$, instead of G_j^η, $1 \leqslant j \leqslant \ell(\eta)$.

Arguing as in the proof of Theorem 10.1 it is easily seen that there exists $\eta_0 > 0$ such that for each $\eta \in (0, \eta_0)$ and $1 \leqslant j \leqslant \ell(\eta)$

$$\sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{D}] \geqslant \mu \Sigma_1 \, |B_1|^{\frac{2}{N}} \eta^{-\frac{2}{N}} - |\tilde{\alpha}|_\infty \, \Sigma_1^{\frac{1}{2}} \, |B_1|^{\frac{1}{N}} \eta^{-\frac{1}{N}} + \inf_\Omega \alpha_0 .$$

Therefore, there exists $\eta_1 \in (0, \eta_0)$ such that for each $\eta \in (0, \eta_1)$

$$\sigma_1^m < \min_{1 \leqslant j \leqslant \ell(\eta)} \sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{D}]. \tag{11.24}$$

Without loss of generality we can assume that

$$\sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{D}] \leqslant \sigma_1^{G_{j+1}^\eta}[\mathscr{L}, \mathscr{D}], \qquad 1 \leqslant j \leqslant \ell(\eta) - 1.$$

Now, fix $\eta \in (0, \eta_1)$ and for each $\delta > 0$ and $1 \leqslant i \leqslant m$ consider the open set

$$\Omega_\delta^i := (\Omega_0^i + B_\delta) \cap \Omega.$$

Since $\bar{\Omega}_0^i \cap \bar{\Omega}_0^j = \varnothing$ if $i \neq j$, there exists $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$

$$\bar{\Omega}_\delta^i \cap \bar{\Omega}_\delta^j = \varnothing \qquad \text{if} \quad i \neq j. \tag{11.25}$$

Moreover, thanks to (11.23), there exists $\delta_1 \in (0, \delta_0)$ such that for each $0 < \delta < \delta_1$

$$\left(\bigcup_{j=1}^{\ell(\eta)} \bar{G}_j^\eta \right) \cap \left(\bigcup_{i=1}^{m} \bar{\Omega}_\delta^i \right) = \varnothing. \tag{11.26}$$

Now, consider the potentials

$$V_i := \begin{cases} V & \text{in } \ \bar{\Omega}_\delta^i, \\ 1 & \text{in } \ \bar{\Omega} \setminus \bar{\Omega}_\delta^i, \end{cases} \qquad 1 \leqslant i \leqslant m. \tag{11.27}$$

Thanks to (11.4), (11.25), and (11.26),

$$\Omega \cap \bigcup_{i=1}^{m} (\bar{\Omega}_\delta^i \setminus \bar{\Omega}_0^i) \subset \Omega_+,$$

and hence for each $1 \leqslant i \leqslant m$ the potential V_i is bounded away from zero on any compact subset of Ω_+, since V is bounded away from zero on any compact subset of Ω_+. Let Γ_1^j, $1 \leqslant j \leqslant n_1$, be the components of Γ_1 and for each $1 \leqslant i \leqslant m$ let $\{j_1, ..., j_{p_i}\}$ denote the subset of $\{1, ..., n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_0^i = \varnothing$ if and only if $j \in \{j_1, ..., j_{p_i}\}$. Then, for each $1 \leqslant i \leqslant m$ we have that

$$\partial\Omega_0^i \cap \bigcup_{k=1}^{p_i} \Gamma_1^{j_k} = \varnothing.$$

In particular,

$$\operatorname{dist}\left(\partial\Omega_0^i, \bigcup_{k=1}^{p_i} \Gamma_1^{j_k} \right) > 0, \qquad 1 \leqslant i \leqslant m,$$

and hence there exists $\delta_2 \in (0, \delta_1)$ such that for each $1 \leqslant i \leqslant m$ and $0 < \delta < \delta_2$

$$\left(\bigcup_{k=1}^{p_i} \Gamma_1^{j_k} + B_\delta \right) \cap \bar{\Omega}_\delta^i = \varnothing. \tag{11.28}$$

Fix $\delta \in (0, \delta_2)$. Then it follows from (11.28) that for each $1 \leqslant i \leqslant m$

$$V_i = 1 \qquad \text{in } \left(\bigcup_{k=1}^{p_i} \Gamma_1^{j_k} + B_\delta \right) \cap \bar{\Omega},$$

and hence for each $1 \leqslant i \leqslant m$ the potential V_i is bounded away from zero on any compact subset of

$$\Omega_+ \cup \bigcup_{k=1}^{p_i} \Gamma_1^{j_k}.$$

Therefore,

$$V_i \in \mathscr{A}, \qquad 1 \leqslant i \leqslant m.$$

Note that for each $1 \leqslant i \leqslant m$ the vanishing set associated to V_i is Ω_0^i, which is connected, and that the corresponding K is the empty set, since (11.23) and (11.26) imply

$$K \cap (\bar{\Omega}_\delta^i \cup \Gamma_1) = \emptyset, \qquad 1 \leqslant i \leqslant m.$$

Thus, each of these potentials fits into the abstract framework of Step 1, and therefore for each $\varepsilon > 0$ there exist $\Lambda_1 = \Lambda_1(\varepsilon) > 0$ and m regular functions

$$\Phi_i \colon \bar{\Omega} \to [0, \infty), \qquad 1 \leqslant i \leqslant m,$$

such that

$$\Phi_i(x) > 0, \qquad x \in \Omega, \quad 1 \leqslant i \leqslant m, \tag{11.29}$$

and for each $1 \leqslant i \leqslant m$ the function Φ_i is a strict supersolution of

$$(\mathscr{L} + \lambda V_i - \sigma_1^{\Omega_0^i}[\mathscr{L}, \mathscr{B}(b, \Omega_0^i)] + \varepsilon, \mathscr{B}(b), \Omega)$$

for each $\lambda > \Lambda_1$.

Now, consider the potentials

$$\hat{V}_j := \begin{cases} 0 & \text{in} \quad G_j^\eta, \\ 1 & \text{in} \quad \bar{\Omega} \setminus G_j^\eta, \end{cases} \qquad 1 \leqslant j \leqslant \ell(\eta). \tag{11.30}$$

Fix $1 \leqslant j \leqslant \ell(\eta)$. By construction, the vanishing set of $\hat{V}_j$ is given by G_j^η which is connected and of class $\mathscr{C}^2$. Moreover, thanks to (11.23), $\bar{G}_j^\eta \subset \Omega$. Thus, there exists $\rho > 0$ such that $\hat{V}_j = 1$ in $(\Gamma_1 + B_\rho) \cap \bar{\Omega}$, and hence

$$\hat{V}_j \in \mathscr{A}, \qquad 1 \leqslant j \leqslant \ell(\eta).$$

Thus, each of these potentials fits into the abstract framework of Step 1, and therefore there exist $\Lambda_2 = \Lambda_2(\varepsilon) > 0$ and $\ell(\eta)$ regular functions

$$\hat{\Phi}_j \colon \bar{\Omega} \to [0, \infty), \qquad 1 \leqslant j \leqslant \ell(\eta),$$

such that

$$\hat{\Phi}_j(x) > 0, \qquad x \in \Omega, \quad 1 \leqslant j \leqslant \ell(\eta), \tag{11.31}$$

and for each $1 \leqslant j \leqslant \ell(\eta)$ the function $\hat{\Phi}_j$ is a strict supersolution of

$$(\mathscr{L} + \lambda \hat{V}_j - \sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{B}(b, G_j^\eta)] + \varepsilon, \mathscr{B}(b), \Omega)$$

for each $\lambda > \Lambda_2$. Note that since $\bar{G}_j^\eta \subset \Omega$, $1 \leqslant j \leqslant \ell(\eta)$,

$$\mathscr{B}(b, G_j^\eta) = \mathscr{D}, \qquad 1 \leqslant j \leqslant \ell(\eta),$$

and hence

$$\sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{B}(b, G_j^\eta)] = \sigma_1^{G_j^\eta}[\mathscr{L}, \mathscr{D}], \qquad 1 \leqslant j \leqslant \ell(\eta). \tag{11.32}$$

Let Γ_1^j, $1 \leqslant j \leqslant n_1$, be the components of Γ_1 and let $\{i_1, ..., i_p\}$ denote the subset of $\{1, ..., n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_0 = \emptyset$ if and only if $j \in \{i_1, ..., i_p\}$. Thanks to Definition 11.1(a), Γ_1^j is a component of $\partial\Omega_0$ for each $j \in \{1, ..., n_1\} \setminus \{i_1, ..., i_p\}$. Moreover,

$$\bigcup_{j=1}^{p} \Gamma_1^{i_j} \cap \partial\Omega_0 = \emptyset,$$

and hence

$$\operatorname{dist}\left(\bigcup_{j=1}^{p} \Gamma_1^{i_j}, \partial\Omega_0\right) > 0. \tag{11.33}$$

Now, consider the δ-neighborhoods defined by (11.12) and (11.13). Thanks to (11.20), (11.23), and (11.33), there exists $\delta_3 \in (0, \delta_2)$ such that for each $0 < \delta < \delta_3$

$$\left(\bigcup_{j=1}^{p} \bar{\mathscr{N}}_\delta^{1, i_j} \cup \bigcup_{j=1}^{n_0} \bar{\mathscr{N}}_\delta^{0, j}\right) \cap \left(\bigcup_{j=1}^{m} \bar{\Omega}_\delta^j \cup \bigcup_{j=1}^{\ell(\eta)} \bar{G}_j^\eta\right) = \emptyset. \tag{11.34}$$

Moreover, since $\Gamma_k^j \cap \Gamma_\ell^i = \emptyset$ if $(i, \ell) \neq (j, k)$, there exists $\delta_4 \in (0, \delta_3)$ such that for each $0 < \delta < \delta_4$

$$\bar{\mathscr{N}}_\delta^{k,j} \cap \bar{\mathscr{N}}_\delta^{\ell,i} = \emptyset \qquad \text{if} \quad (i, \ell) \neq (j, k), \quad k, \ell \in \{0, 1\}. \tag{11.35}$$

Furthermore, since $\lim_{\delta \searrow 0} |\mathscr{N}_\delta^{0,j}| = 0$ for each $1 \leqslant j \leqslant n_0$, it follows from Theorem 10.1 that there exists $\delta_5 \in (0, \delta_4)$ such that for each $0 < \delta < \delta_5$

$$\sigma_1^{\mathscr{N}_\delta^{0,j}}[\mathscr{L}, \mathscr{D}] > \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)] := \sigma_1^0, \qquad 1 \leqslant j \leqslant n_0. \tag{11.36}$$

Now, let ψ^i_δ, $i \in \{i_1, ..., i_p\}$, and ξ^j_δ, $1 \leqslant j \leqslant n_0$, denote the principal eigenfunctions associated to $\sigma_1^{\mathcal{N}^{1,i}_\delta}[\mathcal{L}, \mathcal{B}(b, \mathcal{N}^{1,i}_\delta)]$, $i \in \{i_1, ..., i_p\}$, and $\sigma_1^{\mathcal{N}^{0,j}_\delta}[\mathcal{L}, \mathcal{D}]$, $1 \leqslant j \leqslant n_0$, respectively. Thanks to (11.26), (11.34), and (11.35), the following function is well defined

$$\Phi := \begin{cases} \Phi_i & \text{in } \; \Omega^i_\delta, \quad 1 \leqslant i \leqslant m, \\ \hat{\Phi}_j & \text{in } \; G^\eta_j, \quad 1 \leqslant j \leqslant \ell(\eta), \\ \psi^{i_j}_\delta & \text{in } \; \bar{\mathcal{N}}^{1,i_j}_{\frac{\delta}{2}}, \quad 1 \leqslant j \leqslant p, \\ \xi^j_\delta & \text{in } \; \bar{\mathcal{N}}^{0,j}_{\frac{\delta}{2}}, \quad 1 \leqslant j \leqslant n_0, \\ \zeta_\delta & \text{in } \; \bar{\Omega} \Big\backslash \left(\bigcup_{i=1}^m \Omega^i_\delta \cup \bigcup_{j=1}^{\ell(\eta)} G^\eta_j \cup \bigcup_{j=1}^p \bar{\mathcal{N}}^{1,i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0,j}_{\frac{\delta}{2}} \right), \end{cases} \tag{11.37}$$

where ζ_δ is any positive and regular extension of

$$\bigcup_{i=1}^m \Phi_i \cup \bigcup_{j=1}^{\ell(\eta)} \hat{\Phi}_j \cup \bigcup_{j=1}^p \psi^{i_j}_\delta \cup \bigcup_{j=1}^{n_0} \xi^j_\delta$$

from

$$\bigcup_{i=1}^m \Omega^i_\delta \cup \bigcup_{j=1}^{\ell(\eta)} G^\eta_j \cup \bigcup_{j=1}^p \bar{\mathcal{N}}^{1,i_j}_{\frac{\delta}{2}} \cup \bigcup_{j=1}^{n_0} \bar{\mathcal{N}}^{0,j}_{\frac{\delta}{2}}$$

to $\bar{\Omega}$ which is bounded away from zero. It exists since, thanks to (11.23), (11.29) and (11.31), the functions

$$\Phi_i|_{\partial\Omega^i_\delta \setminus \Gamma_1}, \qquad \hat{\Phi}_j|_{\partial G^\eta_j}, \qquad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant \ell(\eta),$$

are positive and bounded away from zero, as well as the functions

$$\psi^{i_j}_\delta|_{\partial\mathcal{N}^{1,i_j}_{\delta/2} \setminus \Gamma_1}, \qquad \xi^i_\delta|_{\partial\mathcal{N}^{0,i}_{\delta/2} \setminus \Gamma_0}, \qquad 1 \leqslant j \leqslant p, \quad 1 \leqslant i \leqslant n_0.$$

As in Step 1, in the definition of $\Phi(x)$ we should delete the $\psi^{i_j}_\delta$'s if $\Gamma_1 \subset \partial\Omega_0$.

For each $1 \leqslant i \leqslant m$ it follows from the definition of Φ_i that in Ω^i_δ the following relations are satisfied for any $\lambda > \Lambda_1$

$$\begin{aligned} &(\mathcal{L} + \lambda V - \sigma_1^{\Omega_0}[\mathcal{L}, \mathcal{B}(b, \Omega_0)] + \varepsilon)\, \Phi \\ &\qquad = (\mathcal{L} + \lambda V_i - \sigma^1_1 + \varepsilon)\, \Phi_i \geqslant (\mathcal{L} + \lambda V_i - \sigma^i_1 + \varepsilon)\, \Phi_i \geqslant 0, \end{aligned}$$

since $V = V_i$ and

$$\sigma_1^{\Omega_0}[\mathcal{L}, \mathcal{B}(b, \Omega_0)] = \sigma^1_1 \leqslant \sigma^i_1,$$

whereas, thanks to (11.24), for each $1 \leqslant j \leqslant \ell(\eta)$ it follows from the definition of $\hat{\Phi}_j$ that in G_j^η the following relations are satisfied for any $\lambda > \Lambda_2$

$$\begin{aligned}(\mathscr{L}+\lambda V-\sigma_1^{\Omega_0}[\mathscr{L},\mathscr{B}(b,\Omega_0)]+\varepsilon)\,\Phi \\ \geqslant (\mathscr{L}+\lambda\hat{V}_j-\sigma_1^1+\varepsilon)\,\hat{\Phi}_j > (\mathscr{L}+\lambda\hat{V}_j-\sigma_1^{G_j^\eta}[\mathscr{L},\mathscr{D}]+\varepsilon)\,\hat{\Phi}_j \geqslant 0,\end{aligned}$$

since $V \geqslant \hat{V}_j = 0$ in G_j^η.

Thanks to Definition 11.1(b), V is positive and bounded away from zero on any compact subset of

$$\Omega_+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j},$$

and hence there exists $\omega > 0$ such that

$$V \geqslant \omega > 0 \qquad \text{in} \quad \bigcup_{j=1}^{p} \mathscr{N}_{\frac{\delta}{2}}^{1,i_j}.$$

Thus, for each $1 \leqslant j \leqslant p$ in $\mathscr{N}_{\frac{\delta}{2}}^{1,i_j}$ we have that for any $\lambda > 0$

$$\begin{aligned}(\mathscr{L}+\lambda V-\sigma_1^{\Omega_0}[\mathscr{L},\mathscr{B}(b,\Omega_0)]+\varepsilon)\,\Phi \\ \geqslant (\sigma_1^{\mathscr{N}_\delta^{1,i_j}}[\mathscr{L},\mathscr{B}(b,\mathscr{N}_\delta^{1,i_j})]-\sigma_1^1+\varepsilon+\lambda\omega)\,\psi_\delta^{i_j} > 0\end{aligned}$$

if λ is sufficiently large, whereas due to (11.36) for each $1 \leqslant j \leqslant n_0$ it follows from the definition of Φ and ξ_δ^j that in $\mathscr{N}_{\frac{\delta}{2}}^{0,j}$ for any $\lambda > 0$ we have that

$$(\mathscr{L}+\lambda V-\sigma_1^{\Omega_0}[\mathscr{L},\mathscr{B}(b,\Omega_0)]+\varepsilon)\,\Phi \geqslant (\sigma_1^{\mathscr{N}_\delta^{0,j}}[\mathscr{L},\mathscr{D}]-\sigma_1^1+\varepsilon)\,\xi_\delta^j > 0.$$

In

$$\bar{\Omega} \Big\backslash \left(\bigcup_{i=1}^{m} \Omega_\delta^i \cup \bigcup_{j=1}^{\ell(\eta)} G_j^\eta \cup \bigcup_{j=1}^{p} \bar{\mathscr{N}}_{\frac{\delta}{2}}^{1,i_j} \cup \bigcup_{j=1}^{n_0} \bar{\mathscr{N}}_{\frac{\delta}{2}}^{0,j}\right)$$

we have that

$$(\mathscr{L}+\lambda V-\sigma_1^0+\varepsilon)\,\Phi = (\mathscr{L}+\lambda V-\sigma_1^0+\varepsilon)\,\zeta_\delta \geqslant (\mathscr{L}-\sigma_1^0+\varepsilon)\,\zeta_\delta+\lambda\omega\zeta_\delta > 0$$

if $\lambda > 0$ is sufficiently large, since $(\mathscr{L}-\sigma_1^0+\varepsilon)\,\zeta_\delta$ is independent of λ and ζ_δ is bounded away from zero.

Finally,

$$\begin{aligned}\mathscr{B}(b)\,\Phi = \mathscr{D}\Phi = \xi_\delta^j = 0 \qquad &\text{on} \quad \Gamma_0^j, \quad 1 \leqslant j \leqslant n_0, \\ \mathscr{B}(b)\,\Phi = (\partial_\nu + b)\,\Phi = (\partial_\nu + b)\,\psi_\delta^{i_j} = 0 \qquad &\text{on} \quad \Gamma_1^{i_j}, \quad 1 \leqslant j \leqslant p,\end{aligned}$$

and for each $1 \leqslant j \leqslant m$ such that $\partial\Omega_0^j \cap \Gamma_1 \neq \emptyset$ we have that

$$\mathscr{B}(b)\,\Phi = (\partial_\nu + b)\,\Phi = (\partial_\nu + b)\,\Phi_j \geqslant 0 \qquad \text{on} \quad \partial\Omega_\delta^j \cap \Gamma_1 = \partial\Omega_0^j \cap \Gamma_1.$$

Thus

$$\mathscr{B}(b)\,\Phi \geqslant 0 \qquad \text{on} \quad \partial\Omega,$$

and therefore the function Φ defined by (11.37) provides us with a positive strict supersolution of (11.9) of $\lambda > 0$ is sufficiently large. This completes the proof of the theorem when condition (11.20) is satisfied.

Now, suppose

$$\Gamma_0 \cap (\partial\Omega_0 \cup K) \neq \emptyset. \tag{11.38}$$

Let Γ_0^j, $1 \leqslant j \leqslant n_0$, be the components of Γ_0, let $\{i_1, ..., i_q\}$ denote the subset of $\{1, ..., n_0\}$ for which

$$\Gamma_0^j \cap (\partial\Omega_0 \cup K) \neq \emptyset$$

if and only if $j \in \{i_1, ..., i_q\}$, and for each $\eta > 0$ sufficiently small consider the open set

$$\tilde{\Omega} := G_\eta := \Omega \cup \left(\bigcup_{j=1}^{q} \Gamma_0^{i_j} + B_\eta \right).$$

The remaining of the proof consists in constructing $\tilde{\mathscr{L}}$, $\tilde{V}$ and $\tilde{\mathscr{B}}(b)$, as in the proof of Step 1 for the case when condition (11.18) is satisfied, so that $\tilde{\Omega}_0 = \Omega_0$, $\tilde{K} = K$, and

$$\tilde{\Gamma}_0 \cap (\partial\tilde{\Omega}_0 \cup \tilde{K}) = \emptyset.$$

Arguing as in the proof of the second part of Step 1, but this time using the result of Step 2 under condition (11.20), instead of the result of Step 1 under (11.10), covers the current situation. This completes the proof of the theorem. ∎

12. PRINCIPAL EIGENVALUES FOR WEIGHTED PROBLEMS

In this section we use the theory developed in the previous ones to analyze the existence and multiplicity of principal eigenvalues for the linear weighted boundary value problem

$$\begin{cases} \mathscr{L}\varphi = \lambda W \varphi & \text{in} \quad \Omega, \\ \mathscr{B}(b)\,\varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{12.1}$$

where $\lambda \in \mathbf{R}$ and $W \in L_\infty(\Omega)$. A principal eigenvalue is any value of λ for which the problem possesses a positive solution φ. Thus, by the uniqueness of the principal eigenpair associated to

$$(\mathscr{L} - \lambda W, \mathscr{B}(b), \Omega),$$

the principal eigenvalues of (12.1) are given by the zeros of the map

$$\Sigma(\lambda) := \sigma_1^\Omega[\mathscr{L} - \lambda W, \mathscr{B}(b)], \qquad \lambda \in \mathbf{R}. \tag{12.2}$$

The following result provides us with some general properties of $\Sigma(\lambda)$.

THEOREM 12.1. *The map $\Sigma(\lambda)$ defined by* (12.2) *satisfies the following properties:*

(a) *$\Sigma(\lambda)$ is real holomorphic and concave. Therefore, either $\Sigma''(\lambda) = 0$ for any $\lambda \in \mathbf{R}$, or there exists a discrete set $Z \subset \mathbf{R}$ such that $\Sigma''(\lambda) < 0$ for each $\lambda \in \mathbf{R} \setminus Z$. By discrete it is meant that $Z \cap K$ is finite for any compact subset K of $\mathbf{R}$.*

(b) *Assume that there exists an open subset $D_+ \subset \Omega$ for which $\inf_{D_+} W > 0$. Then,*

$$\lim_{\lambda \nearrow \infty} \Sigma(\lambda) = -\infty. \tag{12.3}$$

If in addition $W \geqslant 0$ in Ω, then $\Sigma'(\lambda) < 0$ for each $\lambda \in \mathbf{R}$.

(c) *Assume that there exists an open subset $D_- \subset \Omega$ for which $\sup_{D_-} W < 0$. Then,*

$$\lim_{\lambda \searrow -\infty} \Sigma(\lambda) = -\infty. \tag{12.4}$$

If in addition $W \leqslant 0$ in Ω, then $\Sigma'(\lambda) > 0$ for each $\lambda \in \mathbf{R}$.

(d) *Assume that there exist two open subsets D_+ and D_- of Ω for which*

$$\inf_{D_+} W > 0, \qquad \sup_{D_-} W < 0. \tag{12.5}$$

Then, conditions (12.3) *and* (12.4) *are satisfied. In particular, there exists $\lambda_0 \in \mathbf{R}$ for which*

$$\Sigma(\lambda_0) = \sup_{\lambda \in \mathbf{R}} \Sigma(\lambda).$$

Moreover, $\Sigma'(\lambda_0) = 0$, $\Sigma'(\lambda) > 0$ if $\lambda < \lambda_0$, and $\Sigma'(\lambda) < 0$ if $\lambda > \lambda_0$. Therefore, λ_0 is unique.

Remark 12.2. In Theorem 12.1(a) the first option might occurs. Indeed, if W is constant, then

$$\Sigma(\lambda)=\sigma_1^{\Omega}[\mathscr{L},\mathscr{B}(b)]-\lambda W.$$

Thus, $\Sigma'(\lambda)=-W$ and $\Sigma''(\lambda)=0$ for each $\lambda\in\mathbf{R}$.

Proof of Theorem 12.1. (a) Set

$$\mathscr{L}(\lambda):=\mathscr{L}-\lambda W.$$

If $\mathscr{L}(\lambda)$, $\lambda\in\mathbf{R}$, is regarded as a family of closed operators with common domain $D(\mathscr{L}(\lambda))=W^2_{\mathscr{B}(b)}(\Omega)$ and values in $L^2(\Omega)$, then it is real holomorphic of type (A) in the sense of T. Kato (cf. [26, Chapt. VII, Sect. 2]). Indeed, for all $v\in L^2(\Omega)$ and $u\in W^2_{\mathscr{B}(b)}(\Omega)$, the L_2-product $\int_\Omega v\mathscr{L}(\lambda)\,u$ is real holomorphic in λ. Therefore, we find from [26, Theorems 1.7, 1.8, of Chapt. VII, Sect. 1.3] that $\Sigma(\lambda)$ is real holomorphic in λ. Moreover, if $\varphi(\lambda)$ stands for the principal eigenfunction associated to $\Sigma(\lambda)$, normalized so that $\int_\Omega \varphi^2(\lambda)=1$, then the map

$$\begin{aligned}\mathbf{R}&\longmapsto L_2(\Omega)\\ \lambda&\longrightarrow \varphi(\lambda)\end{aligned}\tag{12.6}$$

is real holomorphic as well. The concavity of the map $\lambda\to\Sigma(\lambda)$ is a consequence from Theorem 5.1. From these features it is easily seen that $\Sigma''(\lambda)\leqslant 0$ for each $\lambda\in\mathbf{R}$. Finally, since $\Sigma''(\lambda)$ is real holomorphic, it follows from the identity theorem that some of the following options occurs: Either $\Sigma''=0$, or Σ'' vanishes in a discrete set, possibly empty. This completes the proof of Part (a).

(b) Assume that there exists an open subset $D_+\subset\Omega$ for which $\inf_{D_+}W>0$. Let $x_+\in D_+$ and $R>0$ such that $B_R(x_+)\subset D_+$. Then, Proposition 3.2 implies

$$\Sigma(\lambda)=\sigma_1^{\Omega}[\mathscr{L}-\lambda W,\mathscr{B}(b)]\leqslant\sigma_1^{B_R(x_+)}[\mathscr{L}-\lambda W,\mathscr{D}],$$

since

$$\mathscr{B}(b,B_R(x_+))=\mathscr{D}\,.$$

Thus, for each $\lambda>0$ we find that

$$\Sigma(\lambda)\leqslant\sigma_1^{B_R(x_+)}[\mathscr{L},\mathscr{D}]-\lambda\inf_{D_+}W,$$

and therefore

$$\lim_{\lambda \nearrow \infty} \Sigma(\lambda) = -\infty.$$

This shows (12.3). Suppose $W \geqslant 0$. Then, it follows from Proposition 3.3 that $\Sigma'(\lambda) \leqslant 0$ for each $\lambda \in \mathbf{R}$. Assume that there exists $\lambda_1 \in \mathbf{R}$ such that

$$\Sigma'(\lambda_1) = 0.$$

Then for each $\lambda < \lambda_1$ we have that

$$0 \geqslant \Sigma'(\lambda) = \int_{\lambda_1}^{\lambda} \Sigma''(s)\, ds \geqslant 0,$$

since $\Sigma'' \leqslant 0$, and hence $\Sigma' = 0$ in $(-\infty, \lambda_1]$. Thus, it follows from the identity theorem that $\Sigma' = 0$ in $\mathbf{R}$, and therefore Σ must be constant. This contradicts (12.3) and shows that $\Sigma'(\lambda) < 0$ for any $\lambda \in \mathbf{R}$.

(c) It readily follows applying Part (b) to the new function

$$\tilde{\Sigma}(\lambda) := \Sigma(-\lambda), \qquad \lambda \in \mathbf{R},$$

associated to the potential $-W$.

(d) It suffices to show that $\Sigma'(\lambda) > 0$ if $\lambda < \lambda_0$, while $\Sigma'(\lambda) < 0$ if $\lambda > \lambda_0$. By definition,

$$\Sigma'(\lambda_0) = 0.$$

Assume that there exists $\lambda_1 < \lambda_0$ such that

$$\Sigma'(\lambda_1) \leqslant 0.$$

Then

$$0 \geqslant \Sigma'(\lambda_1) = \int_{\lambda_0}^{\lambda_1} \Sigma''(\lambda)\, d\lambda \geqslant 0,$$

since $\Sigma'' \leqslant 0$, and hence

$$\Sigma'(\lambda_1) = \int_{\lambda_0}^{\lambda_1} \Sigma''(\lambda)\, d\lambda = 0.$$

Thus, $\Sigma''=0$ in $[\lambda_1, \lambda_0]$ and so, thanks to the identity theorem, we find that $\Sigma''=0$ in $\mathbf{R}$. Hence, there exist two constants a, $b \in \mathbf{R}$ such that

$$\Sigma(\lambda)=a\lambda+b, \qquad \lambda \in \mathbf{R}.$$

This fact contradicts (12.5). Therefore, $\Sigma'(\lambda)>0$ for all $\lambda<\lambda_0$.

Now, assume that there exists $\lambda_2>\lambda_0$ such that

$$\Sigma'(\lambda_2)\geqslant 0.$$

Then

$$0 \leqslant \Sigma'(\lambda_2)=\int_{\lambda_0}^{\lambda_2} \Sigma''(\lambda)\, d\lambda \leqslant 0,$$

and hence

$$\Sigma'(\lambda_2)=\int_{\lambda_0}^{\lambda_2} \Sigma''(\lambda)\, d\lambda=0.$$

Thus, $\Sigma''=0$ in $[\lambda_0, \lambda_2]$ and so, thanks to the identity theorem, we find that $\Sigma''=0$ in $\mathbf{R}$. Hence, there exist two constants a, $b \in \mathbf{R}$ such that

$$\Sigma(\lambda)=a\lambda+b, \qquad \lambda \in \mathbf{R}.$$

This fact contradicts (12.5). Therefore, $\Sigma'(\lambda)<0$ for all $\lambda>\lambda_0$. This completes the proof of the theorem. ∎

From Theorem 12.1 it is easily found the following characterization of the existence of principal eigenvalues for the linear weighted boundary value problem (12.1).

THEOREM 12.3. *The following assertions are true:*

(a) *Assume that* $W \geqslant 0$ *and that there exists an open subset* $D_+ \subset \Omega$ *such that* $\inf_{D_+} W>0$. *Then,* (12.1) *possesses a principal eigenvalue if, and only if,*

$$\lim_{\lambda \searrow -\infty} \Sigma(\lambda)>0. \tag{12.7}$$

Moreover, it is unique if it exists. Let λ_1 *denote it. Then,* λ_1 *is a simple eigenvalue of* $(\mathscr{L}-\lambda_1 W, W)$ *in the sense of* [12].

(b) *Assume that* $W \leqslant 0$ *and that there exists an open subset* $D_- \subset \Omega$ *such that* $\sup_{D_-} W < 0$. *Then,* (12.1) *possesses a principal eigenvalue if, and only if,*

$$\lim_{\lambda \nearrow \infty} \Sigma(\lambda) > 0. \tag{12.8}$$

Moreover, it is unique if it exists. Let λ_1 *denote it. Then,* λ_1 *is a simple eigenvalue of* $(\mathscr{L} - \lambda_1 W, W)$ *in the sense of* [12].

(c) *Assume that there exist two open subsets* D_+ *and* D_- *of* Ω *for which* (12.5) *is satisfied, and let* $\lambda_0 \in \mathbf{R}$ *be the unique value of* λ *for which* $\Sigma'(\lambda) = 0$. *Then,* (12.1) *possesses a principal eigenvalue if, and only if,* $\Sigma(\lambda_0) \geqslant 0$. *Moreover,* λ_0 *is the unique principal eigenvalue of* (12.1) *if* $\Sigma(\lambda_0) = 0$, *whereas* (12.1) *possesses exactly two principal eigenvalues, say* $\lambda_- < \lambda_+$, *if* $\Sigma(\lambda_0) > 0$. *Moreover, in this case*

$$\lambda_- < \lambda_0 < \lambda_+,$$

and $\lambda_\pm$ *is a simple eigenvalue of* $(\mathscr{L} - \lambda_\pm W, W)$ *in the sense of* [12].

Proof. The existence and multiplicity results are an straightforward consequence from Theorem 12.1. It remains to show the simplicity of any eigenvalue, say λ_1, of (12.1) when $\Sigma'(\lambda_1) \neq 0$.

As in the proof of Theorem 12.1, let $\varphi(\lambda)$ denote the principal eigenfunction associated to $\Sigma(\lambda)$, normalized so that $\int_\Omega \varphi^2(\lambda) = 1$. Then, we already know that it is real holomorphic, and hence differentiating

$$(\mathscr{L} - \lambda W)\, \varphi(\lambda) = \Sigma(\lambda)\, \varphi(\lambda)$$

with respect to λ gives

$$(\mathscr{L} - \lambda W)\, \varphi'(\lambda) - W \varphi(\lambda) = \Sigma'(\lambda)\, \varphi(\lambda) + \Sigma(\lambda)\, \varphi'(\lambda).$$

Thus, setting

$$\varphi_1 := \varphi(\lambda_1), \qquad \varphi_1' := \varphi'(\lambda_1),$$

we find that

$$(\mathscr{L} - \lambda_1 W)\, \varphi_1' = W \varphi_1 + \Sigma'(\lambda_1)\, \varphi_1, \tag{12.9}$$

since, by definition, $\Sigma(\lambda_1) = 0$.

Recall that λ_1 is a simple eigenvalue of $(\mathscr{L} - \lambda_1 W, W)$ if, and only if,

$$W \varphi_1 \notin R[\mathscr{L} - \lambda_1 W]. \tag{12.10}$$

Suppose $\Sigma'(\lambda_1) \neq 0$ and $W\varphi_1 \in R[\mathscr{L} - \lambda_1 W]$. Then it follows from (12.9) that

$$\Sigma'(\lambda_1)\,\varphi_1 \in R[\mathscr{L} - \lambda_1 W],$$

and hence

$$\varphi_1 \in R[\mathscr{L} - \lambda_1 W],$$

which is impossible, since

$$N[\mathscr{L} - \lambda_1 W] = \mathrm{span}[\varphi_1]$$

and $\Sigma(\lambda_1) = 0$ is an algebraically simple eigenvalue of $\mathscr{L} - \lambda_1 W$. Therefore, condition (12.10) holds. Conversely, it follows from (12.9) that condition (12.10) fails if $\Sigma'(\lambda_1) = 0$. This completes the proof of the theorem. ▌

Remark 12.4. (a) Under the assumptions of Theorem 12.3(a), $\lambda_1 > 0$ if and only if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0$, and $\lambda_1 = 0$ if and only if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] = 0$.

(b) Under the assumptions of Theorem 12.3(b), $\lambda_1 > 0$ if and only if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < 0$, and $\lambda_1 = 0$ if and only if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] = 0$.

(c) Under the assumptions of Theorem 12.3(c), assume in addition that $\Sigma(\lambda_0) = 0$. Then, $\lambda_0 > 0$ if and only if $\Sigma'(0) > 0$, whereas $\lambda_0 = 0$ if and only if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] = 0$. Now, suppose $\Sigma(\lambda_0) > 0$. Then, $\lambda_- < 0 < \lambda_+$ if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0$, $0 = \lambda_- < \lambda_+$ if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] = 0$ and $\Sigma'(0) > 0$, $\lambda_- < \lambda_+ = 0$ if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] = 0$ and $\Sigma'(0) < 0$, $0 < \lambda_- < \lambda_+$ if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < 0$ and $\Sigma'(0) > 0$, and $\lambda_- < \lambda_+ < 0$ if $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] < 0$ and $\Sigma'(0) < 0$.

(d) Theorem 12.3 is substantially sharper than the classical result of Hess and Kato [22] (cf. [21] as well), where beside the fact that the coefficients of the differential operator and the boundary operator are more restrictive than the current ones, it was required that $\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0$

The following result provides us with some sufficient conditions in terms of the weight function W so that (12.1) can possess a principal eigenvalue. First we shall consider the case of a sign definite potential. Then we shall consider the general case.

THEOREM 12.5. *Suppose $W \in L_\infty(\Omega)$ and $W \geqslant 0$. Then, the following assertions are true:*

(a) *If* $\inf_\Omega W > 0$, *then* (12.1) *possesses a unique principal eigenvalue.*

(b) *If* $\inf_\Omega W = 0$, *W is admissible in the sense of Definition* 11.1, (7.1) *is satisfied and* (2.7) *holds on* $\Gamma_1 \cap \partial\Omega_0$, *then* (12.1) *possesses a principal eigenvalue if, and only if,*

$$\sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)] > 0. \tag{12.11}$$

Moreover, it is unique if it exists.

Similarly, (12.1) *possesses a unique principal eigenvalue if* $\sup_\Omega W < 0$, *whereas when* $\sup_\Omega W = 0$, $-W$ *is admissible in the sense of Definition* 11.1, (7.1) *is satisfied and* (2.7) *holds on* $\Gamma_1 \cap \partial\Omega_0$, *then* (12.1) *possesses a principal eigenvalue if, and only if,* (12.11) *holds. Moreover, it is unique if it exists.*

Proof. (a) Suppose $\inf_\Omega W > 0$. Then for any $\lambda < 0$ we have that

$$\Sigma(\lambda) = \sigma_1^\Omega[\mathscr{L} - \lambda W, \mathscr{B}(b)] \geqslant \sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] - \lambda \inf_\Omega W,$$

and hence

$$\lim_{\lambda \searrow -\infty} \Sigma(\lambda) = \infty.$$

Therefore, (12.7) holds and Theorem 12.3(a) completes the proof.

(b) Suppose $\inf_\Omega W = 0$, $W \in \mathscr{A}$, (7.1), and (2.7) holds on $\Gamma_1 \cap \partial\Omega_0$. Then, thanks to Theorem 11.4,

$$\lim_{\lambda \searrow -\infty} \Sigma(\lambda) = \sigma_1^{\Omega_0}[\mathscr{L}, \mathscr{B}(b, \Omega_0)].$$

Thus, (12.7) occurs if, and only if, (12.11) is satisfied. Thanks to Theorem 12.3(a), this completes the proof if $W \geqslant 0$.

If $W \leqslant 0$, instead of $W \geqslant 0$, it suffices to apply the result just obtained to the new function

$$\tilde{\Sigma}(\lambda) := \Sigma(-\lambda), \qquad \lambda \in \mathbf{R},$$

associated to the potential $-W$. ∎

When W changes of sign and

$$\sigma_1^\Omega[\mathscr{L}, \mathscr{B}(b)] > 0,$$

it follows from Theorem 12.3(c) that (12.1) possesses two principal eigenvalues; one negative and the other positive. If $(\mathscr{L}, \mathscr{B}(b), \Omega)$ does not satisfy the strong maximum principle, then the following results are satisfied.

THEOREM 12.6. *Suppose* (7.1) *and*

$$\sigma_1^{\Omega}[\mathscr{L}, \mathscr{B}(b)] < 0. \tag{12.12}$$

Let $W \in L_\infty(\Omega)$ *be a potential for which there exist two open subsets* D_+, $D_- \subset \Omega$ *such that condition* (12.5) *is satisfied. Set*

$$W^+ := \max\{W, 0\}, \qquad W^- := W^+ - W,$$

and assume that some of the following two conditions is satisfied:

(a) $W^+ \in \mathscr{A}$, *the class of admissible potentials in* Ω, (2.7) *is satisfied on* $\Gamma_1 \cap \partial\Omega_0^+$, *and*

$$\sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{B}(b, \Omega_0^+)] > 0, \qquad \|W^-\|_{L_\infty(\Omega)} \leqslant \max_{\lambda \leqslant \lambda_1^+} \frac{\Sigma_+(\lambda)}{-\lambda}, \tag{12.13}$$

where Ω_0^+ *stands for the vanishing open set associated to* W^+, *whose existence is guaranteed by Definition* 11.1,

$$\Sigma_+(\lambda) := \sigma_1^{\Omega}[\mathscr{L} - \lambda W^+, \mathscr{B}(b)], \qquad \lambda \in \mathbf{R},$$

and $\lambda_1^+ < 0$ *stands for the unique* λ *for which* $\Sigma_+(\lambda) = 0$.

(b) $W^- \in \mathscr{A}$, *the class of admissible potentials in* Ω, (2.7) *is satisfied on* $\Gamma_1 \cap \partial\Omega_0^-$, *and*

$$\sigma_1^{\Omega_0^-}[\mathscr{L}, \mathscr{B}(b, \Omega_0^-)] > 0, \qquad \|W^+\|_{L_\infty(\Omega)} \leqslant \max_{\lambda \geqslant \lambda_1^-} \frac{\Sigma_-(\lambda)}{\lambda}, \tag{12.14}$$

where Ω_0^- *stands for the vanishing open set associated to* W^-, *whose existence is guaranteed by Definition* 11.1,

$$\Sigma_-(\lambda) := \sigma_1^{\Omega}[\mathscr{L} + \lambda W^-, \mathscr{B}(b)], \qquad \lambda \in \mathbf{R},$$

and $\lambda_1^- > 0$ *stands for the unique* λ *for which* $\Sigma_-(\lambda) = 0$.

Then, (12.1) *possesses exactly two principal eigenvalues. Moreover, in case* (a) *the two principal eigenvalues are negative, whereas in case* (b) *the two principal eigenvalues of* (12.1) *are positive.*

Remark 12.7. Thanks to Corollary 10.3, (12.13) is satisfied if $|\Omega_0^+|$ is sufficiently small, b is sufficiently large, and W^- is sufficiently small, while (12.14) holds if $|\Omega_0^-|$ is sufficiently small, b is sufficiently large and W^+ is sufficiently small. In fact, thanks to Theorem 10.1, if $\Gamma_1 \cap \partial\Omega_0^+ = \emptyset$, and $|\Omega_0^+|$ and W^- are sufficiently small, then the restriction on b is unnecessary to have (12.13). Similarly, if $\Gamma_1 \cap \partial\Omega_0^- = \emptyset$, and $|\Omega_0^-|$ and W^+ are sufficiently small, then the restriction on b is unnecessary to have (12.14).

Proof of Theorem 12.6. First we show that (12.13) and (12.14) make sense. Suppose

$$\sigma_1^{\Omega_0^+}[\mathcal{L}, \mathcal{B}(b, \Omega_0^+)] > 0.$$

Then, the existence and the uniqueness of λ_1^+ is guaranteed by Theorem 12.5(b). Moreover, $\lambda_1^+ < 0$ and, thanks to Theorem 12.1(b),

$$\Sigma'_+(\lambda_1^+) < 0.$$

Thus,

$$\lim_{\lambda \nearrow \lambda_1^+} \frac{\Sigma_+(\lambda)}{-\lambda} = 0.$$

On the other hand, for each $\lambda < \lambda_1^+$ we have that

$$\frac{\Sigma_+(\lambda)}{-\lambda} > 0,$$

and due to Theorem 11.4

$$\lim_{\lambda \searrow -\infty} \frac{\Sigma_+(\lambda)}{-\lambda} = 0.$$

Therefore,

$$\max_{\lambda \leqslant \lambda_1^+} \frac{\Sigma_+(\lambda)}{-\lambda} \in (0, \infty)$$

is well defined. In particular, (12.13) makes sense. Similarly, it is easily seen that (12.14) makes sense.

Suppose condition (a) is satisfied. Then, there exists $\tilde{\lambda} < \lambda_1^+ < 0$ such that

$$\|W^-\|_{L_\infty(\Omega)} \leqslant \frac{\Sigma_+(\tilde{\lambda})}{-\tilde{\lambda}},$$

and hence

$$\sigma_1^\Omega[\mathcal{L} - \tilde{\lambda} W^+, \mathcal{B}(b)] \geqslant -\tilde{\lambda}\, \|W^-\|_{L_\infty(\Omega)}.$$

Thus,

$$\Sigma(\tilde{\lambda}) = \sigma_1^\Omega[\mathcal{L} - \tilde{\lambda} W, \mathcal{B}(b)] > \sigma_1^\Omega[\mathcal{L} - \tilde{\lambda} W^+, \mathcal{B}(b)] + \tilde{\lambda}\, \|W^-\|_{L_\infty(\Omega)} \geqslant 0,$$

since $\tilde{\lambda} < 0$ and $W^- < \|W^-\|_{L_\infty(\Omega)}$. Theorem 12.3(c) completes the proof of this theorem when condition (a) holds. The previous argument can be easily adapted to prove the theorem when condition (b), instead of (a), is satisfied. This concludes the proof. ∎

THEOREM 12.8. *Suppose* (7.1) *and*

$$\sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)] = 0. \tag{12.15}$$

Let $W \in L_\infty(\Omega)$ be a potential for which there exist two open subsets D_+, $D_- \subset \Omega$ such that condition (12.5) *is satisfied. Set*

$$W^+ := \max\{W, 0\}, \qquad W^- := W^+ - W,$$

and assume that some of the following two conditions is satisfied:

(a) *$W^+ \in \mathcal{A}$, the class of admissible potentials in Ω,* (2.7) *is satisfied on $\Gamma_1 \cap \partial\Omega_0^+$, and*

$$\|W^-\|_{L_\infty(\Omega)} < \max_{\lambda \leqslant 0} \frac{\Sigma_+(\lambda)}{-\lambda}, \tag{12.16}$$

where Ω_0^+ stands for the vanishing open set associated to W^+ and

$$\Sigma_+(\lambda) := \sigma_1^\Omega[\mathcal{L} - \lambda W^+, \mathcal{B}(b)], \qquad \lambda \in \mathbf{R}.$$

(b) *$W^- \in \mathcal{A}$, the class of admissible potentials in Ω,* (2.7) *is satisfied on $\Gamma_1 \cap \partial\Omega_0^-$, and*

$$\|W^+\|_{L_\infty(\Omega)} < \max_{\lambda \geqslant 0} \frac{\Sigma_-(\lambda)}{\lambda}, \tag{12.17}$$

where Ω_0^- stands for the vanishing open set associated to W^- and

$$\Sigma_-(\lambda) := \sigma_1^\Omega[\mathcal{L} + \lambda W^-, \mathcal{B}(b)], \qquad \lambda \in \mathbf{R}.$$

Then, (12.1) *possesses exactly two principal eigenvalues. Moreover, in case* (a) *one of them is zero and the other is negative, whereas in case* (b) *one of them is zero and the other is positive.*

Proof. Fist we show that (12.16) and (12.17) make sense. Thanks to (12.15),

$$\Sigma_+(0) = \sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)] = 0,$$

and hence due to the monotonicity of the principal eigenvalue with respect to the domain

$$\sigma_1^{\Omega_0^+}[\mathscr{L}, \mathscr{B}(b, \Omega_0^+)] > 0.$$

Moreover, thanks to Theorem 12.1(b),

$$\Sigma'_+(0) < 0.$$

Thus

$$\lim_{\lambda \nearrow 0} \frac{\Sigma_+(\lambda)}{-\lambda} = -\Sigma'_+(0) > 0.$$

On the other hand, for each $\lambda < 0$ we have that

$$\frac{\Sigma_+(\lambda)}{-\lambda} > 0,$$

and due to Theorem 11.4

$$\lim_{\lambda \searrow -\infty} \frac{\Sigma_+(\lambda)}{-\lambda} = 0.$$

Therefore,

$$\max_{\lambda \leqslant 0} \frac{\Sigma_+(\lambda)}{-\lambda} \in (0, \infty)$$

is well defined. In particular, (12.16) makes sense. Similarly, it is easily seen that (12.17) makes sense.

Suppose that condition (a) is satisfied. Then, there exists $\tilde{\lambda} < 0$ such that

$$\|W^-\|_{L_\infty(\Omega)} < \frac{\Sigma_+(\tilde{\lambda})}{-\tilde{\lambda}},$$

and hence

$$\sigma_1^\Omega[\mathscr{L} - \tilde{\lambda} W^+, \mathscr{B}(b)] > -\tilde{\lambda}\, \|W^-\|_{L_\infty(\Omega)}.$$

Thus,

$$\Sigma(\tilde{\lambda}) = \sigma_1^\Omega[\mathscr{L} - \tilde{\lambda} W, \mathscr{B}(b)] > \sigma_1^\Omega[\mathscr{L} - \tilde{\lambda} W^+, \mathscr{B}(b)] + \tilde{\lambda}\, \|W^-\|_{L_\infty(\Omega)} > 0,$$

since $\tilde{\lambda} < 0$. Using Theorem 12.3(c) completes the proof of this theorem when condition (a) is satisfied. The previous argument can be easily adapted to prove the theorem when condition (b), instead of (a), is satisfied. This concludes the proof. ∎

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